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Ergodicity of Quantum Cellular Automata

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We define a class of dynamical maps on the quasi-local algebra of a quantum spin system, which are quantum analoges of probabilistic cellular automata. We develop criteria for such a system to be ergodic, i.e., to possess a unique invariant state. Intuitively, ergodicity obtains if the local transition operators exhibit sufficiently large disorder. The ergodicity criteria also imply bounds for the exponential decay of correlations in the unique invariant state. The main technical tool is a quantum version of oscillation norms, defined in the classical case as the sum over all sites of the variations of an observable with respect to local spin flips.

KEY WORDS: Cellular automata; interacting particle systems; quantum spin systems; approach to equilibrium; oscillation norm.

1. INTRODUCTION

A probabilistic cellular automaton (PCA),[1-3] or interacting particle system,[4] can be regarded as an infinite collection of cells or particles, where each cell or particle can take on a finite number of states. The discrete time evolution of such systems is determined by a statistical law according to which, in any given configuration at time t, all cells are simultaneously and independently updated to the configuration at time $t+1$. One of the basic questions concerning such systems is ergodicity, i.e., the uniqueness of the stationary probability measure on configurations. In this article we will introduce a quantum analog of this structure, and we will also prove an analog of well-known criteria by which ergodicity of a PCA can be decided in terms of the local transition probabilities.[4, 2, 1]

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Our definition of quantum cellular automata (QCA) is an abstraction of a structure that arose in the project of generalizing the construction of "finitely correlated states" on quantum spin chains[5,6] to two-dimensional systems.[7] In this context the ergodicity of the QCA is equivalent to the state on the two-dimensional system being uniquely determined by local data, independent of boundary conditions. The term "quantum cellular automaton" has been used previously by some other authors.[8–10] In the cases we are aware of, however, it is used for a structure on the Hilbert space level, and not on the level of observables. Thus in ref. 8 the classical states at each site are simply replaced by the values of the wave function at that site, and the dynamics is just a discrete Schrödinger equation with non-self-adjoint Hamiltonian, made nonlinear by keeping the normalization fixed. There is some interest in quantum cellular automata also from the point of view of nanometer-scale computers, for which quantum effects are expected to be relevant.[11,12,10] The evolution of the automata considered in this paper is in general nonunitary, i.e., pure states may evolve into mixed states. This might be an interesting addition to the structure of "quantum computers", as studied by a number of authors recently (see ref. 13, and references cited therein).

The main technical contribution of this article is the introduction of a general class of "oscillation norms" on quantum lattice systems. We believe this to be a useful tool of independent interest, and therefore include proofs of the basic general properties of such norms. They generalize a classical notion, which also was a principal tool for the proof of the ergodicity criteria in refs. 2, 1, and 14. A special case of such a norm in the quantum case was already used extensively by Matsui.[15–18] Among other things, he used it to establish ergodicity criteria for the continuous-time analog of QCAs. Another special case of oscillation norms, used for the same purpose, is to be found in a recent preprint by Majewski and Zegarlinski.[19]

An important consideration in the study of transition operators on composite quantum systems is that the norm of an operator acting on observables may increase if we consider the given system as a subsystem of a larger one. It is therefore essential to consider versions of the basic operator properties which are "stabilized with respect to system enlargement." The stabilized versions of positivity and boundedness (called "complete" positivity and boundedness) are well known, and we give a brief summary of these with references in an appendix. For the notion of "boundedness in oscillation norm" the stabilized version is described in Section 4. Much to our surprise, it turned out that in this case the necessity of stabilization is not characteristic of the quantum case. Even in the classical case, as soon as one has more than two states at each site, the oscillation norm bound of a transition operator may increase with the size of the environment, as we will show by an explicit example.

The paper is organized as follows. In Section 2 we establish our notations for quantum lattice systems and develop the definition of QCAs from the classical notion of PCAs. Section 3.1 introduces oscillation norms in general C^*-algebras, and shows how contractivity of a transition operator in such a norm entails ergodicity. Section 3.2 is devoted to the construction of a canonical oscillation norm for a system composed of many parts with given oscillation norms, such as the quasi-local algebra. The key result, reducing estimates on the infinite systems to an estimate of a local quantity, is shown in Section 4.1: if the local transition operators contract with a certain rate, as measured by the "completely bounded oscillation norm," then the same holds for an infinite tensor product of such operators. We briefly indicate in Section 4.2 how other plausible approaches to such estimates fail on this account. The basic ergodicity criterion for QCAs in Section 5.1 is a direct corollary of this estimate. In Section 5.2 we show in what sense the classical PCAs are covered by this criterion. Finally, we show in Section 5.3 that the same estimate on local transition operators which implies ergodicity by our general criterion also entails exponential decay of correlation functions in the unique invariant state. A quick review of the notions of complete positivity and complete boundedness, with pointers to the literature, is given in the appendix.

2. DEFINITION OF QUANTUM CELLULAR AUTOMATA

In order to describe the notion of quantum cellular automata (QCA), it is best to begin by restating the classical structure in an algebraic language more suitable for generalization to the quantum case. The underlying lattice, or set of cells, will be denoted by $\mathscr{L}$. To each cell $x \in \mathscr{L}$ we associate an observable algebra $\mathscr{A}^x$, which in the case of a classical system is simply the algebra $\mathscr{A}^x = \mathscr{C}(\Omega^x)$ of continuous complex-valued functions on the set Ω^x of configurations of each cell. Finite subsystems, associated with finite subsets $\Lambda \subset \mathscr{L}$, are described by the tensor product

$$\mathscr{A}^\Lambda = \bigotimes_{x \in \Lambda} \mathscr{A}^x = \mathscr{C}\left(\mathop{\times}_{x \in \Lambda} \Omega^x \right) \tag{2.1}$$

This formula is also used for infinite subsystems, and, in particular, for the observable algebra $\mathscr{A}^{\mathscr{L}}$ of the whole system. The product on the right-hand side is then the infinite Cartesian product of compact topological spaces. Even if each cell has only finitely many configurations, and hence "continuity" for functions on each Ω^x is a vacuous condition, continuity of the observables $f \in \mathscr{A}^{\mathscr{L}}$ is the nontrivial requirement that f may be uniformly approximated by observables depending only on finitely many cells.

The PCA dynamics is determined by the transition functions

$$p_x(\omega, \sigma) = \mathbb{E}\{\omega_{x,t+1} \in \sigma \mid \omega_t = \omega\}$$

for $x \in \mathscr{L}$, $\omega \in \Omega$, and measurable $\sigma \subset \Omega^x$. This is a probability measure on Ω^x in its second argument, depending, in principle, on the entire previous configuration $\omega \in \Omega$. In algebraic language this becomes an operator

$$\begin{aligned} P_x\colon \quad & \mathscr{A}^x \to \mathscr{A}^{\mathscr{L}} \\ (P_x f)(\omega) &= \int p_x(\omega, d\omega_x)\, f(\omega_x) \end{aligned} \tag{2.2}$$

The assumption that $P_x f$ is a continuous function is called the Feller property of the PCA.[18,4] It means intuitively that the updating of one cell does not depend too sensitively on infinitely many other cells. It is automatically satisfied for finite-range interactions, i.e., when $P_x(\mathscr{A}^x) \subset \mathscr{A}^{\Lambda(x)} \subset \mathscr{A}^{\mathscr{L}}$, for some finite set $\Lambda(x)$.

Independent updating of different cells, the basic property of PCAs, means in more formal language that the distribution of ω_{t+1}, given ω_t, is the product measure formed out of the measures $p_x(\omega_t, \cdot)$. Equivalently, the transition operator $P\colon \mathscr{A}^{\mathscr{L}} \to \mathscr{A}^{\mathscr{L}}$ is defined as

$$P\left(\bigotimes_{x \in \Lambda} f^x\right) = \prod_{x \in \Lambda} P_x(f^x) \tag{2.3}$$

where $f^x \in \mathscr{A}^x$, $\Lambda \subset \mathscr{L}$, is an arbitrary finite set, and the product on the right-hand side is the pointwise product of functions in the algebra $\mathscr{A}^{\mathscr{L}} = \mathscr{C}(\Omega)$. Since the tensor products on the left hand side of (2.3) span the C^*-algebra $\mathscr{C}(\Omega)$, this equation determines P uniquely. This concludes our brief description of PCA dynamics.

Some of the above is easily translated into the *quantum setting*: the main change is that now all observable algebras may be noncommutative (rather than commutative) C^*-algebras with identity. For the observable algebra of a single cell one typically chooses the algebra $\mathscr{M}_n$ of $n \times n$ matrices with $n < \infty$. The simplest example is a Heisenberg spin-1/2 system, for which $\mathscr{A}^x = \mathscr{M}_2$ for every $x \in \mathscr{L}$. The observable algebra of a composite quantum system is defined as the closure of the algebraic tensor product in a suitable C^*-norm. In contrast to the classical case there may be several such norms, in which case we always take the "minimal" C^*-norm.[20] For the product of finite-dimensional matrix algebras, in which we will mostly be interested, all C^*-tensor norms coincide anyhow. We continue to use the notation $\mathscr{A}^{\Lambda}$ for the observable algebras (2.1) of finite regions. For $\Lambda_1 \subset \Lambda_2$, there is a natural inclusion $\mathscr{A}^{\Lambda_1} \subset \mathscr{A}^{\Lambda_2}$, by tensoring each element of $\mathscr{A}^{\Lambda_1}$ with the identity in $\mathscr{A}^{\Lambda_2/\Lambda_1}$. The *infinite tensor product* defining the observable algebra of an infinite (sub-)system is always defined as the C^*-inductive limit[21] of the finite tensor products with respect to these inclusions. As in the classical case, this simply means that all observables can be approximated in norm by finitely localized ones. A *transition operator* such as P_x can be characterized as a positive operator (i.e., an operator taking positive elements into positive elements) mapping the identity into the identity. Moreover, we will also assume that these properties persist if we consider $\mathscr{A}^x$ and $\mathscr{A}^{\mathscr{L}}$ as subsystems of a larger system, i.e., we require P_x to be completely positive (see the appendix for definitions). A probability measure is replaced in the algebraic framework by the expectation value functional or *state* it induces. Thus states are linear functionals on the observable algebra which take positive values on positive elements and the value 1 on the identity.

The key problem for the quantum generalization of PCAs is the positivity of the right-hand side of Eq. (2.3): in the noncommutative context a product of positive elements is practically never positive. In fact, restricting to the case of just two factors ($\Lambda = \{x, y\}$ with $f^x, f^y \geq 0$), we find that a necessary and sufficient condition for the positivity of P is that the ranges of the operators P_x commute in $\mathscr{A}^{\mathscr{L}}$ [see Proposition IV.4.23(ii) in ref. 20].

We therefore have to choose our definition in such a way that the commuting range condition holds automatically. The following is one way of doing this. It is perhaps not the most general possibility, but it covers the cellular automata which came up naturally in our construction of states on two-dimensional spin systems. The idea is to subdivide each cell into subcells, such that the images of different P_x are contained in different tensor factors with respect to the refined tensor decomposition of $\mathscr{A}^{\mathscr{L}}$. For notational convenience we state the definition only in the case that the observable algebras and transition operators of different cells are all isomorphic.

Definition 1. *A quantum cellular automaton* (QCA) is given by the following objects:

1. A countable lattice $\mathscr{L}$.
2. A set S of "subcell types" and a C^*-algebra $\mathscr{B}^s$ with unit, for each type $s \in S$.

3. For each $s \in S$, an injective "propagation map" $d(s; \cdot): \mathscr{L} \to \mathscr{L}$.
4. A completely positive unit-preserving operator $P_1: \bigotimes_{s \in S} \mathscr{B}^s \to \bigotimes_{s \in S} \mathscr{B}^s$.

The following objects are defined in terms of the above:

5. At each site $x \in \mathscr{L}$ the algebra $\mathscr{A}^x = \bigotimes_{s \in S} \mathscr{B}^{x,s}$, where $\mathscr{B}^{x,s}$ is an isomorphic copy of $\mathscr{B}^s$.
6. The quasi-local algebra $\mathscr{A}^{\mathscr{L}} = \bigotimes_{x \in \mathscr{L}} \mathscr{A}^x$.
7. The transition operators

$$P_x: \quad \mathscr{A}^x = \bigotimes_{s \in S} \mathscr{B}^{x,s} \to \bigotimes_{s \in S} \mathscr{B}^{d(s;x),s} \subset \mathscr{A}^{\mathscr{L}}$$

   where P_x is defined from P_1 by identifying the tensor factor $\mathscr{B}^s$ in the range of P_1 with $\mathscr{B}^{d(s;x),s} \subset \mathscr{A}^{d(s;x)}$.

8. The total transition operator $P: \mathscr{A}^{\mathscr{L}} \to \mathscr{A}^{\mathscr{L}}$ defined by Eq. (2.3).

To see that the operator in step 8 is well defined, observe that the subcells are labeled by $\mathscr{L} \times S$. and that the sets $R^x = \{(d(s;x), s) \mid s \in S\} \subset \mathscr{L} \times S$ which describe the range of P_x are disjoint. Then, by Proposition IV.4.23(i) in ref. 20, P is completely positive. Since $P\mathbb{1} = \mathbb{1}$, this implies that P is norm continuous, and consequently has a unique extension by continuity to the whole quasi-local algebra $\mathscr{A}^{\mathscr{L}}$.

Of course, when we think of a lattice, the injective maps $d(s; \cdot)$ will typically be lattice translations. If we choose all $d(s; \cdot)$ to be the identity, we obtain a system of noninteracting cells $\mathscr{A}^x$. Note that the subcell decomposition is only relevant in the range of P_x, not in the domain. Thus in each step the subcell decomposition of the previous step is obliterated. In particular, the second iterate P^2 of a QCA cannot be written in the same form: the algebras $P^2(\mathscr{A}^x)$ do not commute with each other. This does not contradict the necessity of the commuting range condition explained above, because the product form of P^2 is also lost. Note that this is not an artefact of our quantum generalization: even in the classical case the second-generation updates of a PCA are no longer independent, hence they no longer satisfy the definition of a PCA.

The adjoint operator of P takes states into states, and we will usually denote its action by $\omega \mapsto \omega \circ P$. It is easy to see (e.g., using the Markov–Kakutani theorem, Theorem V.10.6 in ref. 22) that any QCA has an *invariant state*, i.e., a state ρ such that $\rho \circ P = \rho$. A QCA is called *ergodic* if there is only one invariant state for P. The main problem addressed in this paper is to find sufficient criteria for ergodicity in terms of the given local data P_1 and $d(s; \cdot)$. We are also interested in stronger versions of ergodicity, e.g., the property that P^n contracts in norm to the invariant state, i.e.,

$$\lim_{n \to \infty} \|P^n(A) - \rho(A)\, \mathbb{1}\| = 0$$

for all $A \in \mathscr{A}^{\mathscr{L}}$. A further closely related problem is to estimate the decay of correlations in the invariant state.

To see what is involved, it is good to look at the most trivial example: a noninteracting particle system. Then we have only one type of subcell and $d(1; \cdot) = \mathrm{id}_{\mathscr{L}}$. Here P is simply the infinite tensor product of copies P_x of a fixed operator P_1 acting on isomorphic finite-dimensional algebras $\mathscr{A}^x$. It is obvious that the restriction of an invariant state for P_1 to a single site is invariant for P_1, and, conversely, any product state formed out of invariant one-site states will be invariant for P (the latter construction need not be exhaustive). Hence P is ergodic if and only if P_1 is ergodic. It is plausible that contractivity properties should also carry over. Assume, for example, that

$$\|P_1^n(A) - \rho_1(A)\, \mathbb{1}\| \leqslant \varepsilon^n\, \|A - \rho_1(A)\, \mathbb{1}\|$$

for all $A \in \mathscr{A}^x$, and ρ_1 the unique invariant state of P_1. Does this imply a similar bound for P? A direct estimate gives indeed a similar bound, but with ε^n multiplied by the number of sites (see Section 4.2). Hence this approach is not feasible on an infinite lattice. What one needs is a norm such that a contractivity estimate for a tensor product of completely positive operators P_x is not worse than the maximum of the estimates for the factors. This is precisely the role of the oscillation norms used in the classical results of ref. 2. Their quantum analog will be studied in the following two sections. We will then return to QCAs in Section 5.

3. OSCILLATION NORMS ON C^*-ALGEBRAS

3.1. Definition and Basic Properties

In this section we want to generalize the notion of oscillation norm to the noncommutative setting. The basic idea remains the same: we consider some operations δ_α which annihilate constants, i.e., $\delta_\alpha(\mathbb{1}) = 0$. In the classical case these operations measure the effect of the spin flips at different sites, i.e.,

$$(\delta_\alpha f)(\sigma_\alpha, \sigma_{\mathrm{not}\,\alpha}) = \tfrac{1}{2}[f(-\sigma_\alpha, \sigma_{\mathrm{not}\,\alpha}) - f(\sigma_\alpha, \sigma_{\mathrm{not}\,\alpha})] \qquad (3.1)$$

where $\sigma_{\text{not}\,\alpha}$ stands for all spin variables at sites other than α. Then we can say that an observable f is nearly constant if the "oscillation norm" $\sum_\alpha \|\delta_\alpha(f)\|$ is small for all α.

For classical systems with more than two states per cell, as well as for any quantum system, there are many ways of "flipping" a single cell. There are several proposals in the literature for how to take this into account. However, all proposals agree that the "total oscillation norm" of a lattice system should be the sum of the oscillations of each cell. Moreover, we will see below that the total oscillation norms in these different approaches are equivalent whenever the local cells are described by finite-dimensional algebras. Perhaps the simplest proposal[19] is to define the oscillation of an observable A localized in a single cell as

$$|||A|||_0 = \|A - \eta(A)\,\mathbb{1}\| \tag{3.2}$$

where η is the normalized trace (or any other state) on the cell algebra $\mathscr{A}^x$. Other approaches use a family $\{\delta_\alpha\}$ of operators for each cell, and one can consider the "sup-oscillation norm"

$$\overline{|||A|||} = \sup_\alpha \|\delta_\alpha(A)\| \tag{3.3}$$

It is also suggestive to define

$$|||A|||_d = \sup_{\eta,\eta'} \frac{|\eta(A) - \eta'(A)|}{d(\eta, \eta')} \tag{3.4}$$

where the supremum is over all pairs of states, and d is some metric on the state space. In the classical case one could restrict the supremum to pure states, so that $|||A|||_d$ is just the Lipshitz constant of A with respect to the metric d.[23] Finally, one may use for the single cell precisely the same form as for the total oscillation, namely a sum

$$|||A||| := \sum_\alpha \|\delta_\alpha(A)\| \tag{3.5}$$

over "elementary" oscillations $\delta_\alpha(A)$. This is the approach used by Matsui.[15] We will also adopt it, mainly because it agrees best with the subcell structure of cellular automata: the propagation maps will introduce a reshuffling of subcells between different main cells, and this process preserves oscillation norms only if the oscillation within each cell is defined by the same mechanism as the total oscillation. This approach also simplifies the presentation in the sense that we can use the same results about oscillation norms for the single cells as well as for the whole system.

Definition 2. Let $\mathscr{A}$ be a C^*-algebra, $\mathbb{1} \in \mathscr{A}$. Let I be an index set and $\{\delta_\alpha | \alpha \in I\}$ a collection of bounded linear operators $\delta_\alpha = \mathscr{A} \to \mathscr{A}$ such that

1. $\delta_\alpha(\mathbb{1}) = 0, \forall \alpha \in I.$
2. $\mathscr{A}_{\text{fin}} := \{A \in \mathscr{A} \mid \sum_{\alpha \in I} \|\delta_\alpha(A)\| < \infty\}$ is $\|\cdot\|$-dense in $\mathscr{A}$.
3. There exists a state $\eta \in \mathscr{A}^*$ (reference state) such that for $A \in \mathscr{A}_{\text{fin}}$

$$\|A - \eta(A)\,\mathbb{1}\| \leqslant \sum_{\alpha \in I} \|\delta_\alpha(A)\| \tag{3.6}$$

Then, for $A \in \mathscr{A}$,

$$|||A||| := \sum_{\alpha \in I} \|\delta_\alpha(A)\|$$

is called the *oscillation norm* of $A \in \mathscr{A}$.

Obviously, $|||\cdot|||$ is seminorm on $\mathscr{A}$. It satisfies

$$\inf_{\lambda \in \mathbb{C}} \|A - \lambda\mathbb{1}\| \leqslant \|A - \eta(A)\,\mathbb{1}\| \leqslant |||A||| \tag{3.7}$$

In particular, $|||A||| = 0$ implies that A is a multiple of the identity.

A simple argument shows that an infimum such as the one on the left-hand side of (3.7) is attained at some $\lambda = \lambda(A)$ in any normed space $\mathscr{A}$ with a fixed element $\mathbb{1} \in \mathscr{A}$. In a Hilbert space we can even assert that $\lambda(A)$ is uniquely determined and depends linearly on A. In a C^*-algebra, however, the shape of the unit ball is different and although, for Hermitian A, $\lambda(A)$ is uniquely determined, it is a nonlinear functional. If the overall bound in (3.7) holds, however, we can easily make *every* state an admissible reference state, albeit for the oscillation norm $|||A|||' = 2|||A|||$: if λ is such that $\|A - \lambda\mathbb{1}\| \leqslant |||A|||$, then, for any state η,

$$\|A - \eta(A)\,\mathbb{1}\| = \|(A - \lambda\mathbb{1}) + \eta(A - \lambda\mathbb{1})\,\mathbb{1}\| \leqslant 2\,\|A - \lambda\mathbb{1}\| \leqslant 2\,|||A|||$$

This shows that the choice of the reference state is largely arbitrary. However, the existence of some such state is a nontrivial constraint on $\{\delta_\alpha\}$, as the following example shows.

Example 3. Let $\mathscr{A} = \bigotimes_{i \in \mathbb{Z}} \mathscr{A}^i$ be the quasi-local algebra on the one-dimensional chain, with $\mathscr{A}^i \cong \mathscr{A}^1$ for all i. Denote by $\tau: \mathscr{A} \to \mathscr{A}$ the translation automorphism, defined by

$$\tau\left(\bigotimes_{i \in \mathbb{Z}} A^i\right) = \bigotimes_{i \in \mathbb{Z}} A^{i+1}$$

We define an operator $\delta: \mathscr{A} \to \mathscr{A}$ by

$$\delta(A) := \tau(A) - A$$

Then $\delta(A) = 0$ is equivalent to the translation invariance of A, which in the quasi-local algebra $\mathscr{A}$ is equivalent to $A \in \mathbb{C}\mathbb{1}$. Thus the one-element collection $\{\delta\}$ satisfies the first two conditions of Definition 2 (with $\mathscr{A}_{\text{fin}} = \mathscr{A}$), and, moreover, $\delta(A) = 0$ implies $A \in \mathbb{C}\mathbb{1}$. But condition 3 is violated in the strong form that there is no finite constant C such that (3.7) holds as $\|A - \lambda\mathbb{1}\| \leqslant C\, |||A|||$.

To see this, pick some element $A_1 \in \mathscr{A}^1 \backslash \mathbb{C}\mathbb{1}$ in the one-site algebra and a function $f: \mathbb{Z} \to \mathbb{R}$ with $f(i) \geqslant 0$, and $\sum_i f(i) = 1$. Then set

$$A = \sum_i f(i)\, \tau^i(A_1)$$

Let ω_1 be a state on $\mathscr{A}^1$, and let $\omega_1^{\mathbb{Z}} = \bigotimes_{i \in \mathbb{Z}} \omega_1$ be the infinite product state on $\mathscr{A}$. Then

$$\|A - \lambda\mathbb{1}\| \geqslant \omega_1^{\mathbb{Z}}(A - \lambda\mathbb{1}) = \omega_1(A_1) - \lambda$$

$$\|\delta(A)\| = \left\| \sum_i \left[f(i) - f(i-1) \right] \tau^i(A_1) \right\|$$

$$\leqslant \|A_1\| \sum_i |f(i) - f(i-1)|$$

With a suitable choice of ω_1 and λ we find $\|A - \lambda\mathbb{1}\| \geqslant \inf_{\lambda'} \|A_1 - \lambda'\mathbb{1}\|$, and hence, with $|||A||| = \|\delta(A)\|$,

$$C^{-1} \leqslant \frac{|||A|||}{\|A - \lambda\mathbb{1}\|} \leqslant \frac{\|A_1\|}{\inf_{\lambda'} \|A_1 - \lambda'\mathbb{1}\|} \sum_i |f(i) - f(i-1)|$$

By choosing f to be slowly varying, e.g., $f(i) = (1-\mu)/(1+\mu)\mu^{|i|}$, for $\mu \to I$, the right hand side can be made arbitrarily small. △

Oscillation norms defined with a single δ_α are precisely those of the form (3.2), i.e., $\delta(A) = A - \eta(A)\mathbb{1}$. The lower bound (3.6) is then equivalent to $|||A|||_0 \leqslant |||A|||$. In the other direction we have the estimate (see Appendix IV of ref. 19)

$$|||A||| = \sum_\alpha \|\delta_\alpha(A - \eta(A)\,\mathbb{1})\| \leqslant \left(\sum_\alpha \|\delta_\alpha\| \right) |||A|||_0 \tag{3.8}$$

Hence, if there are finitely many oscillation operators δ_α, or the above sum is otherwise convergent, the norms $|||\cdot|||$ and $|||\cdot|||_0$ are equivalent. This is not a big surprise, since on a finite-dimensional $\mathscr{A}$, all seminorms which vanish exactly on the constants are equivalent.

In the next example we will consider a special case of Definition 2, which will sometimes be especially convenient. The oscillation norms used in refs. 15–17 and 19 are of this form.

Example 4. We say that the operators δ_α satisfy *Matsui's condition* when

$$\sum_{\alpha \in I} \delta_\alpha(A) = A - \eta(A)\,\mathbb{1} \tag{3.9}$$

for some state η. Then, by a simple application of the triangle inequality, condition 3 of Definition 2 is satisfied. Another situation in which this condition comes up naturally is the following: Let $\mathscr{A} = \mathscr{M}_n$ be the algebra of $n \times n$ matrices, and let G be a finite group. Consider an irreducible projective representation $g \mapsto U_g \in \mathscr{M}_n$ of G, and set

$$\delta_g(A) = \frac{1}{|G|}\,(A - U_g A U_g^*) \tag{3.10}$$

Then $|G|^{-1} \sum_g U_g A U^*_g$ commutes with all U_g, and is hence of the form $\eta(A)\mathbb{1}$. Clearly, η is an invariant state with respect to all U_g, and must therefore be the normalized trace of $\mathscr{M}_n$. Since for the identity element $e \in G$ we have $\delta_e = 0$, it suffices to take the above δ_g for $g \in I = G \backslash \{e\}$. With this choice, Matsui's condition (3.9) holds. Note that since $\|A\| = \|A U_g\|$ the oscillation norm may be written in the suggestive form

$$|||A||| = \frac{1}{|G|} \sum_{g \in G} \|[A, U_g]\|$$

The simplest special case is to take the U_g as the three Pauli matrices $\sigma_\alpha \in \mathscr{M}_2$ and $U_e = \mathbb{1}$: the product of any two of these operators is in the same set, up to a phase. The group $G(|G| = 4)$ consists of the rotations by π around the three Cartesian axes in $\mathbb{R}^3$. In this case one also finds easily that the reference state η is uniquely determined: let η' be another reference state. Then

$$\|\sigma_1 - \eta'(\sigma_1)\,\mathbb{1}\| \leqslant |||\sigma_1||| = \tfrac{1}{4} \sum_{i=1}^{3} \|[\sigma_1, \sigma_i]\| = \tfrac{1}{4}(\|2\sigma_3\| + \|2\sigma_2\|) = 1$$

implies that $\eta'(\sigma_1) = 0$. Repeating the same argument for the other components, we find that $\eta' = \eta$ has to be the normalized trace. △

The infimum on the left-hand side of (3.7) is the standard quotient norm of $\mathscr{A}/\mathbb{C}\mathbb{1}$. Hence $|||\cdot|||$ can be considered as a proper norm on the subspace $\mathscr{A}_{\text{fin}}/\mathbb{C}\mathbb{1}$ of this quotient. We now show that this norm turns $\mathscr{A}_{\text{fin}}/\mathbb{C}\mathbb{1}$ into a Banach space.

Lemma 5. $\mathscr{A}_{\text{fin}}/\mathbb{C}\mathbb{1}$ is $|||\cdot|||$-complete.

Proof. Let $\{A_n\} \subset \mathscr{A}_{\text{fin}}/\mathbb{C}\mathbb{1}$ be a $|||\cdot|||$ Cauchy sequence. It follows that $\|A_n\| \leqslant |||A_n|||$. Because $\mathscr{A}/\mathbb{C}\mathbb{1}$ is complete for the quotient norm $\|\cdot\|$, there exists an $A \in \mathscr{A}/\mathbb{C}\mathbb{1}$ such that $\|A - A_n\| \to 0$ for $n \to \infty$. We have to show that $A \in \mathscr{A}_{\text{fin}}/\mathbb{C}\mathbb{1}$. Let $I' \subset I$ be a finite subset of the index set. Then, because each δ_α is bounded,

$$\begin{aligned}\sum_{\alpha \in I'} \|\delta_\alpha(A)\| &= \lim_{n \to \infty} \sum_{\alpha \in I'} \|\delta_\alpha(A_n)\| \\ &\leqslant \limsup_{n \to \infty} \sum_{\alpha \in I} \|\delta_\alpha(A_n)\| \\ &\leqslant \lim_{n \to \infty} |||A_n|||\end{aligned}$$

This limit exists because A_n is $|||\cdot|||$-Cauchy. Taking the supremum over all finite $I' \subset I$, we find that $|||A||| < \infty$. Similarly, we find that

$$\sum_{\alpha \in I'} \|\delta_\alpha(A - A_n)\| \leqslant \lim_{m \to \infty} |||A_m - A_n|||$$

and since this bound is independent of $I' \subset I$, we have that $\lim_n |||A - A_n||| = 0$. ∎

Since we have not postulated any further properties of the operators δ_α, the space $\mathscr{A}_{\text{fin}}$ does not come with a natural algebraic structure. However, in the special case where $\delta_\alpha(A) = i[A, D_\alpha]$ are derivations, but also in the case (3.10), we get

$$\|\delta_\alpha(AB)\| \leqslant \|\delta_\alpha(A)\| \cdot \|B\| + \|A\| \cdot \|\delta_\alpha(B)\|$$

and hence

$$|||AB||| \leqslant |||A||| \cdot \|B\| + \|A\| \cdot |||B||| \tag{3.11}$$

In particular, $\mathscr{A}_{\text{fin}}$, becomes a Banach algebra with the norm $\|A\|_\lambda = \|A\| + \lambda\, |||A|||$ for any $\lambda > 0$.

The main reason for introducing oscillation norms is that in the case of large systems it is often easier to establish contractivity in this norm than in the C^*-norm $\|\cdot\|$. Nevertheless, as the following proposition shows, this contractivity is sufficient to establish convergence of the iterates in the C^*-norm, and hence ergodicity. Note that while the oscillation norms (3.2)–(3.5) are "equivalent" in finite-dimensional situations, the best constant ε in the assumption of the proposition will depend on the choice of norm.

Proposition 6. Let $\mathscr{A}$ be a C^*-algebra with oscillation norm $|||\cdot|||$, and consider a linear operator $P: \mathscr{A} \to \mathscr{A}$ such that $\|P(A)\| \leqslant \|A\|$, $P(\mathbb{1}) = \mathbb{1}$, and, for some fixed $\varepsilon < 1$, and all $A \in \mathscr{A}_{\text{fin}}$,

$$|||P(A)||| \leqslant \varepsilon\, |||A|||$$

Then there exists a unique state $\rho \in \mathscr{A}^*$, such that $\rho \circ P = \rho$. Moreover:

1. $\lim_{n \to \infty} \|P^n(A) - \rho(A)\mathbb{1}\| = 0$ for all $A \in \mathscr{A}$.
2. $\|P^n(A) - \rho(A)\mathbb{1}\| \leqslant 2\varepsilon^n\, |||A|||$ for all $A \in \mathscr{A}_{\text{fin}}$.

Proof. Fix an element $A \in \mathscr{A}_{\text{fin}}$ and consider the sequence $A_n = P^n(A)$. Choose some numbers λ_n, for example $\lambda_n = \eta(A_n)$, such that

$$\|A_n - \lambda_n \mathbb{1}\| \leqslant |||A_n|||$$

Then $\{\lambda_n\}$ is a Cauchy sequence: taking $n \geqslant m$ without loss of generality, we get

$$\begin{aligned} |\lambda_n - \lambda_m| &\leqslant \|\lambda_n \mathbb{1} - A_n\| + \|A_n - \lambda_m \mathbb{1}\| \\ &\leqslant |||A_n||| + \|P^{n-m}\| \cdot \|A_m - \lambda_m \mathbb{1}\| \\ &\leqslant (\varepsilon^n + \varepsilon^m)\, |||A||| \end{aligned}$$

Let $\tilde{\rho}(A) = \lim_{m \to \infty} \lambda_m$. Inserting this definition into the previous inequality, we find

$$|\lambda_n - \tilde{\rho}(A)| \leqslant \varepsilon^n\, |||A|||$$

and for $A \in \mathscr{A}_{\text{fin}}$, it follows that

$$\begin{aligned} \|P^n(A) - \tilde{\rho}(A)\, \mathbb{1}\| &\leqslant \|P^n(A) - \lambda_n \mathbb{1}\| + |\lambda_n - \tilde{\rho}(A)| \\ &\leqslant |||A_n||| + \varepsilon^n\, |||A||| \\ &\leqslant 2\varepsilon^n\, |||A||| \end{aligned}$$

$\tilde{\rho}(A)$ is linear, because $A \mapsto \tilde{\rho}(A)\mathbb{1}$ is the limit of the linear operators P^n. In addition, $\tilde{\rho}(A)$ is also a $\|\cdot\|$-continuous functional on $\mathscr{A}_{\text{fin}}$:

$$\begin{aligned} |\tilde{\rho}(A)| &\leqslant \|\tilde{\rho}(A)\, \mathbb{1} - P^n(A)\| + \|P^n(A)\| \\ &\leqslant 2\varepsilon^n\, |||A||| + \|A\| \xrightarrow{n \to \infty} \|A\| \end{aligned}$$

Let ρ denote the continuous extension of $\tilde{\rho}$ on $\mathscr{A}$. Then $|\rho(A)| \leqslant \|A\|$, i.e., $\|\rho\| = 1$, and also $\rho(\mathbb{1})\mathbb{1} = \lim_n P^n(\mathbb{1}) = \mathbb{1}$. hence ρ is a state.

For showing the convergence property 1, let $A \in \mathscr{A}$, and $\delta > 0$. Then we may pick $A_\delta \in \mathscr{A}_{\text{fin}}$ with $\|A - A_\delta\| \leqslant \delta$. It follows that

$$\begin{aligned}\|P^n(A) - \rho(A)\,\mathbb{1}\| &\leqslant \|P^n(A - A_\delta)\| + \|P^n(A_\delta) - \rho(A_\delta)\,\mathbb{1}\| + |\rho(A_\delta - A)| \\ &\leqslant \|A - A_\delta\| + 2\varepsilon^n\,|||A_\delta||| + \|A_\delta - A\| \\ &\leqslant 2\delta + 2\varepsilon^n\,|||A_\delta|||\end{aligned}$$

Hence, for sufficiently large n, the left-hand side becomes arbitrarily small.

Finally, suppose $\nu \in \mathscr{A}^*$ is another fixed point of the adjoint of P, i.e., $\nu \circ P = \nu$. Then

$$\nu(A) = \lim_{n \to \infty} (\nu \circ P^n)(A) = \nu(\rho(A)\,\mathbb{1}) = \rho(A) \cdot \nu(\mathbb{1})$$

Hence, if ν is also normalized, we have $\nu = \rho$. ∎

We want to use this criterion to establish ergodicity of cellular automata describing spin systems. The operator P will then be built up from local operators P_x. We thus have to construct an oscillation norm for composite systems, i.e., for tensor products of algebras with oscillation norm, and then have to apply Proposition 6 to the product system.

3.2. Tensorable Oscillation Norms

We now take the algebra $\mathscr{A} = \mathscr{A}^{\mathscr{L}}$ to be the quasi-local algebra over a lattice $\mathscr{L}$, to each site of which is attached a unital C^*-algebra $\mathscr{A}^x$, i.e.,

$$\mathscr{A} \equiv \mathscr{A}^{\mathscr{L}} = \bigotimes_{x \in \mathscr{L}} \mathscr{A}^x$$

Assume now that we are given an oscillation norm for the algebras $\mathscr{A}^x$ at each site. We would like to assemble from this an oscillation norm for $\mathscr{A}^{\mathscr{L}}$.

The basic idea is very simple: as the collection of "flip" operators we simply take the union of the flip operators for each site, i.e.,

$$\bigcup_{x \in \mathscr{L}} \{\delta_\alpha^{(x)} \mid \alpha \in I^x\} \tag{3.12}$$

where $\delta_\alpha^{(x)} = \delta_\alpha^x \otimes \mathrm{id}^{\mathscr{L} \setminus \{x\}}$ is just the action of δ_α^x at site x of the quasilocal algebra. The oscillation norm of an element $A \in \mathscr{A}$ is then

$$|||A||| = \sum_{x \in \mathscr{L}} \sum_{\alpha \in I^x} \|\delta_\alpha^{(x)}(A)\| \tag{3.13}$$

This quantity is always defined, but possibly infinite. So we can define, as before,

$$\mathscr{A}_{\text{fin}} = \{A \in \mathscr{A} \mid |||A||| < \infty\}$$

Clearly, the finite tensor products $\bigotimes_{x \in \Lambda} A_x$, with $A_x \in \mathscr{A}^x_{\text{fin}}$, and their linear combinations are in $\mathscr{A}_{\text{fin}}$. By Definition 2, $\mathscr{A}^x_{\text{fin}} \subset \mathscr{A}^x$ is norm dense, and by definition of the quasi-local algebra the finite tensor products span a dense subspace of $\mathscr{A}$. Hence $\mathscr{A}_{\text{fin}} \cap \mathscr{A}_{\text{loc}}$, and a *fortiori* $\mathscr{A}_{\text{fin}}$, is dense in $\mathscr{A}$. In order for $|||\cdot|||$ to become an oscillation norm, we need to establish the estimate (3.6) in the definition of oscillation norms, with the obvious candidate

$$\eta = \bigotimes_{x \in \mathscr{L}} \eta^x \tag{3.14}$$

for a reference state. For this, we need the following "stabilized version" of (3.6). It is automatically satisfied in the commutative case (cf. Proposition 10 below). For the notion of complete boundedness, see the appendix. When η is a state on the C^*-algebra $\mathscr{A}$, we denote by $\hat{\eta}$: $\mathscr{A} \to \mathscr{A}$ the operator $\hat{\eta}(A) = \eta(A)\mathbb{1}$.

Definition 7. An oscillation norm defined by operators $\{\delta_\alpha \mid \alpha \in I\}$ on a C^*-algebra $\mathscr{A}$, with reference state η, is called *tensorable*, if each δ_α is completely bounded, and

$$\|A - (\text{id}_{\mathscr{M}_n} \otimes \hat{\eta})(A)\| \leqslant \sum_{\alpha \in I} \|(\text{id}_{\mathscr{M}_n} \otimes \delta_\alpha)(A)\| \tag{3.15}$$

for $A \in \mathscr{M}_n(\mathbb{C}) \otimes \mathscr{A}$.

At first sight it may seem rather special to allow tensoring only with $\mathscr{M}_n$. However, as in the definition of complete positivity, this is sufficient to give the analogous statement for all C^*-algebras.

Lemma 8. Let $\{\delta_\alpha | \alpha \in I\}$ define a tensorable oscillation norm on a C^*-algebra $\mathscr{A}$, and let $\mathscr{M}$ be any unital C^*-algebra. Let $\mathscr{M} \otimes \mathscr{A}$ be the minimal C^*-tensor product and $A \in \mathscr{M} \otimes \mathscr{A}$. Then

$$\|A - (\text{id}_{\mathscr{M}} \otimes \hat{\eta})(A)\| \leqslant \sum_{\alpha \in I} \|(\text{id}_{\mathscr{M}} \otimes \delta_\alpha)(A)\| \tag{3.15'}$$

Proof. We need the following basic observation: if p_γ is a net of Hilbert space operators with $\|p_\gamma\| \leqslant 1$, converging strongly to the identity operator then, for any bounded operator A,

$$\lim_\gamma \|p_\gamma^* A p_\gamma\| = \|A\| \tag{3.16}$$

Indeed, the inequality

$$\limsup_{\gamma} \|p_\gamma^* A p_\gamma\| \leqslant \limsup_{\gamma} \|p_\gamma\|^2 \, \|A\| \leqslant \|A\|$$

is trivial. On the other hand, let ϕ, ψ be unit vectors such that $|\langle \phi, A\psi \rangle| \geqslant \|A\| - \varepsilon$, and let γ be sufficiently large such that $\|p_\gamma \phi - \phi\|$, $\|p_\gamma \psi - \psi\| \leqslant \varepsilon$. Then

$$\|p_\gamma^* A p_\gamma\| \geqslant |\langle p_\gamma \phi, A p_\gamma \psi \rangle| \geqslant |\langle \phi, A\psi \rangle| - 2\varepsilon \, \|A\| \geqslant \|A\| - \varepsilon - 2\varepsilon \, \|A\|$$

and (3.16) follows by taking the inferior limit and $\varepsilon \to 0$.

Without loss we may take $\mathscr{M}$ and $\mathscr{A}$ to be faithfully represented on some Hilbert spaces. Then the minimal C^*-tensor product $\mathscr{M} \otimes \mathscr{A}$ is defined as the C^*-algebra generated by operators of the form $M \otimes A$, with $M \in \mathscr{M}$ and $A \in \mathscr{A}$, respectively. Let p_γ denote a net of finite-dimensional projections in the representation space of $\mathscr{M}$, converging to the identity, and introduce the operators $\hat{p}_\gamma(M) = p_\gamma M p_\gamma$. Then, for $A \in \mathscr{M} \otimes \mathscr{A}$, we have $\hat{A} = (\hat{p}_\gamma \otimes \mathrm{id}_{\mathscr{A}})(A) \in \mathscr{M}_n \otimes \mathscr{A}$, where n is the dimension of p_γ. Hence, because $(\hat{p}_\gamma \otimes \mathrm{id}_{\mathscr{A}})$ and $(\mathrm{id}_{\mathscr{M}} \otimes \hat{\eta})$ commute,

$$\begin{aligned}
\|(\hat{p}_\gamma \otimes \mathrm{id}_{\mathscr{A}})(A - (\mathrm{id}_{\mathscr{M}} \otimes \hat{\eta})(A))\| &= \|\hat{A} - (\mathrm{id}_{\mathscr{M}_n} \otimes \hat{\eta})(\hat{A})\| \\
&\leqslant \sum_{\alpha \in I} \|(\mathrm{id}_{\mathscr{M}_n} \otimes \delta_\alpha)(\hat{A})\| \\
&= \sum_{\alpha \in I} \|(\hat{p}_\gamma \otimes \mathrm{id}_{\mathscr{A}})(\mathrm{id}_{\mathscr{M}} \otimes \delta_\alpha)(A)\| \\
&\leqslant \sum_{\alpha \in I} \|(\mathrm{id}_{\mathscr{M}} \otimes \delta_\alpha)(A)\|
\end{aligned}$$

Hence the result follows by applying (3.16) to the left-hand side of this inequality and the family of projections $p_\gamma \otimes \mathbb{1}$. ∎

The basic result concerning the tensor product of algebras with oscillation norm is the following. The second part was proven in a special case by Matsui.[18]

Proposition 9. Let $\{\mathscr{A}^x | x \in \mathscr{L}\}$ be algebras with tensorable oscillation norms, defined by $\{\delta_\alpha^x | \alpha \in I^x\}$, and with reference states $\eta^x \in (\mathscr{A}^x)^*$. Then:

1. $\mathscr{A} = \bigotimes_{x \in \mathscr{L}} \mathscr{A}^x$ is an algebra with tensorable oscillation norm given by (3.12), with reference state (3.14).
2. $\mathscr{A}_{\mathrm{fin}} \cap \mathscr{A}_{\mathrm{loc}}$ is dense in $\mathscr{A}_{\mathrm{fin}}$, with respect to $|||\cdot|||$.

Proof. We show part 2 first. For any finite subset $\Lambda \subset \mathscr{L}$, define

$$\hat{\eta}^{\Lambda^c} := \prod_{x \notin \Lambda} \hat{\eta}^{(x)}$$

where $\hat{\eta}^{(x)}(A) = (\hat{\eta}^x \otimes \mathrm{id}^{\mathscr{L} \setminus \{x\}})(A)$. Therefore the range of $\hat{\eta}^{\Lambda^c}$ is contained in the local algebra $\bigotimes_{x \in \Lambda} \mathscr{A}^x$. For $A \in \mathscr{A}_{\text{loc}}$, we trivially have

$$\|\hat{\eta}^{\Lambda^c}(A) - A\| \xrightarrow{\Lambda \to \mathscr{L}} 0 \qquad (*)$$

because Λ absorbs the localization region of A. Consequently, because $\mathscr{A}_{\text{loc}} \subset \mathscr{A}$ is $\|\cdot\|$-dense, ($*$) is valid for all $A \in \mathscr{A}$.

We show next that

$$|||\hat{\eta}^{\Lambda^c}(A) - A||| \xrightarrow{\Lambda \to \mathscr{L}} 0$$

for $A \in \mathscr{A}_{\text{fin}}$. One has

$$\begin{aligned}
&|||\hat{\eta}^{\Lambda^c}(A) - A||| \\
&\quad = \sum_{x \in \mathscr{L}} \sum_{\alpha \in I^x} \|\delta_\alpha^{(x)}(\hat{\eta}^{\Lambda^c} - \mathrm{id}^{\mathscr{L}})(A)\| \\
&\quad = \sum_{x \in \Lambda} \sum_{\alpha \in I^x} \|\delta_\alpha^{(x)}(\hat{\eta}^{\Lambda^c} - \mathrm{id}^{\mathscr{L}})(A)\| + \sum_{x \notin \Lambda} \sum_{\alpha \in I^x} \|\delta_\alpha^{(x)}(\hat{\eta}^{\Lambda^c} - \mathrm{id}^{\mathscr{L}})(A)\| \\
&\quad = \sum_{x \in \Lambda} \sum_{\alpha \in I^x} \|(\hat{\eta}^{\Lambda^c} - \mathrm{id}^{\mathscr{L}})\, \delta_\alpha^{(x)}(A)\| + \sum_{x \notin \Lambda} \sum_{\alpha \in I^x} \|\delta_\alpha^{(x)}(A)\|
\end{aligned}$$

because for $x \in \Lambda$ the operators $\hat{\eta}^{\Lambda^c}$ and $\delta_\alpha^{(x)}$ commute, and for $x \notin \Lambda$, $\delta_\alpha^{(x)}$ acts on the factor $\mathbb{1}^{(x)}$, so $\delta_\alpha^{(x)} \circ \hat{\eta}^{\Lambda^c} = 0$. In the limit $\Lambda \to \mathscr{L}$ the second term vanishes because the sum over all $x \in \mathscr{L}$ converges for $A \in \mathscr{A}_{\text{fin}}$. The first sum is termwise dominated by the sum defining $2\,|||A|||$. However, because $\delta_\alpha^{(x)}(A) \in \mathscr{A}$, each term in this sum goes to zero by ($*$). Hence the sum goes to zero by dominated convergence. Hence $|||\hat{\eta}^{\Lambda^c}(A) - A||| \to 0$, and $\mathscr{A}_{\text{fin}} \cup \mathscr{A}_{\text{loc}}$ is $|||\cdot|||$-dense in $\mathscr{A}_{\text{fin}}$.

To prove part 1 we have to verify Definition 2. The boundedness of the $\delta_\alpha^{(x)}$ follows from the complete boundedness of the operators $\delta_\alpha^{(x)}$ on $\mathscr{A}^x$. The property $\delta_\alpha^{(x)}(\mathbb{1}) = 0$ is evident, and we argued for the $|||\cdot|||$-density of $\mathscr{A}_{\text{fin}}$, above, before (3.14). Hence only the estimate (3.6) remains to be seen. We begin by showing it for $A \in \mathscr{A}_{\text{fin}} \cap \mathscr{A}_{\text{loc}}$, say A localized in a finite region $\Lambda \subset \mathscr{L}$. We conveniently label the sites in Λ as $1, \ldots, |\Lambda|$. Then with a telescoping sum we find

$$\left\| A - \bigotimes_{x \in \Lambda} \hat{\eta}^x(A) \right\| \leqslant \sum_{k=1}^{|\Lambda|} \left\| \bigotimes_{x<k} \hat{\eta}^x \otimes (\mathrm{id}^k - \hat{\eta}^k) \otimes \bigotimes_{y>k} \mathrm{id}^y(A) \right\|$$
$$= \sum_{k=1}^{|\Lambda|} \left\| \prod_{x<k} \hat{\eta}^{(x)} \right\| \cdot \|A - \hat{\eta}^{(k)}(A)\|$$
$$\leqslant \sum_{k \in \Lambda} \sum_{\alpha \in I^k} \|\delta_\alpha^{(k)}(A)\| = |||A|||$$

where $\hat{\eta}^{(x)}\colon \mathscr{A} \to \mathscr{A}$ is the operator $\hat{\eta}^x \otimes \mathrm{id}^{\mathscr{L} \setminus \{x\}}$. At the last estimate we used that the product of the $\hat{\eta}^{(x)}$ is a completely positive unit-preserving operator, which hence has norm 1, and, of course, the tensorability of $|||\cdot|||$. Hence the required estimate holds for $A \in \mathscr{A}_{\mathrm{fin}} \cap \mathscr{A}_{\mathrm{loc}}$. In particular, it holds for the approximants $\hat{\eta}^{\Lambda^c}(A)$ of a general $A \in \mathscr{A}_{\mathrm{fin}}$. Since by part 2 of the proposition these elements approximate A in both norms, both sides of the estimate converge as $\Lambda \to \mathscr{L}$. This completes the proof that $|||\cdot|||$ is an oscillation norm. It is tensorable, because we can include an additional tensor factor $\mathscr{M}_n$ in the product defining $\mathscr{A}$, and use the same estimates as above to establish (3.15). ▮

With this proposition the construction of oscillation norms for infinite systems is reduced to the construction of tensorable oscillation norms for the one-site algebras. Recall from (3.8) that all oscillation norms for which $\sum_{\alpha \in I} \|\delta_\alpha\| < \infty$ are equivalent. By summing this estimate over all cells, we conclude similarly that all oscillation norms on a composite system for which

$$\sum_{\alpha \in I^x} \|\delta_\alpha^x\|_{\mathrm{cb}} \leqslant c < \infty \tag{3.17}$$

with a constant independent of x are also equivalent. In particular, this holds for the norm in ref. 19. Equivalence can also be shown for global oscillation norms, which are defined as the sum of local oscillations, which are in turn given by some supremum[23] [see (3.3) or (3.40.)].

Showing tensorability is especially easy in the classical case:

Proposition 10. On an Abelian C^*-algebra $\mathscr{A}$, every oscillation norm is tensorable.

Proof. We can set $\mathscr{A} = \mathscr{C}(X)$ for some compact space X. Let $\mathscr{M}$ be any C^*-algebra. Then $\mathscr{M} \otimes \mathscr{A} \cong \mathscr{C}(X, \mathscr{M})$, the algebra of $\mathscr{M}$-valued continuous functions on X, with the norm

$$\|A\| := \sup_x \|A(x)\| = \sup_{x, \Phi} |\Phi(A(x))| \tag{3.18}$$

where the supremum over Φ is with respect to all linear functionals on $\mathcal{M}$ with $\|\Phi\| \leqslant 1$. Then $((\mathrm{id}_{\mathcal{M}} \otimes \delta_\alpha)A)(x) = \int \delta_\alpha(x, dy) A(y)$, where we have used δ_α to denote both the operator and its integral kernel. Applying the estimate (3.6) to the continuous function $x \mapsto \Phi(A(x))$, we obtain

$$\begin{aligned}
\|A - (\mathrm{id}_{\mathcal{M}} \otimes \hat{\eta})(A)\| &= \sup_{x, \Phi} \left| \Phi(A(x)) - \int \eta(dy)\, \Phi(A(y)) \right| \\
&\leqslant \sup_{\Phi} \sum_{\alpha \in I} \sup_{x} \left| \int \delta_\alpha(x, dy)\, \Phi(A(y)) \right| \\
&\leqslant \sum_{\alpha \in I} \sup_{\Phi} \sup_{x} \left| \int \delta_\alpha(x, dy)\, \Phi(A(y)) \right| \\
&= \sum_{\alpha \in I} \|(\mathrm{id}_{\mathcal{M}} \otimes \delta_\alpha)(A)\| \qquad \blacksquare
\end{aligned}$$

In the noncommutative case, tensorability is a nontrivial constraint on oscillation norms. The following is a handy criterion.

Lemma 11. Suppose that the operators $\{\delta_\alpha \mid \in I\}$ defining an oscillation norm $|||\cdot|||$ on a C^*-algebra $\mathcal{A}$ satisfy Matsui's condition (see Example 4 above). Then it is tensorable.

Proof. Let $A \in \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$. Then, by (3.9),

$$\begin{aligned}
\sum_{\alpha \in I} (\mathrm{id}_{\mathcal{M}_n} \otimes \delta_\alpha)(A) &= \left(\mathrm{id}_{\mathcal{M}_n} \otimes \sum_{\alpha \in I} \delta_\alpha \right)(A) \\
&= (\mathrm{id}_{\mathcal{M}_n} \otimes (\mathrm{id}_{\mathcal{A}} - \hat{\eta}))(A) \\
&= A - (\mathrm{id}_{\mathcal{M}_n} \otimes \hat{\eta})(A)
\end{aligned}$$

By taking norms on both sides we find (3.15). $\blacksquare$

Finally, we record for later use that for tensor product operators a version of (3.11) holds without further assumptions: using that in a minimal C^*-algebra tensor product $\|A \otimes B\| = \|A\| \cdot \|B\|$, we find

$$|||A \otimes B||| = |||A||| \cdot \|B\| + \|A\| \cdot |||B||| \tag{3.19}$$

4. CONTRACTIVITY OF TENSOR PRODUCT OPERATORS

4.1. Oscillation Norm Estimate

In the simplest, noninteracting case of a cellular automaton the total transition operator P is the infinite tensor product of the one-site transition operators P_x. If we know that each P_x contracts exponentially with rate $\varepsilon < 1$ to a multiple of the identity, can we also assert this about P? This turns out to be the crucial question for developing ergodicity estimates for quantum cellular automata. We will show in this subsection that, provided we use oscillation norms to express contractivity, the product P indeed contracts with the same rate. In fact the validity of this bound is the main reason for considering oscillation norms. In order to make this point more precise we show in the next subsection how other norms fail to give the desired estimate.

The most straightforward definition of contractivity of a transition operator P is given by the estimate $|||P(A)||| \leqslant \varepsilon\, |||A|||$ as in Proposition 6. The best constant ε in this estimate is conveniently denoted by $|||P|||$; it is the norm of P as an operator on $\mathscr{A}_{\text{fin}}$. However, this quantity is not appropriate for the study of composite systems, since even in the classical case we may have $|||\text{id}_{\mathscr{M}} \otimes P||| > |||P|||$ (cf. the example at the end of this subsection). Therefore, we use a version of $|||P|||$, which is "stabilized" with respect to coupling the system to an outside world.

Definition 12. Let $\mathscr{A}$ and $\mathscr{B}$ be algebras with tensorable oscillation norms generated by $\{\delta_\alpha | \alpha \in I\}$ and $\{\tilde{\delta}_\beta | \beta \in J\}$, respectively. Let $P\colon \mathscr{A} \to \mathscr{B}$ be linear and completely bounded. Then the *completely bounded oscillation norm* of P, denoted by $|||P|||_{\text{cb}}$, is defined as the smallest constant ε for which the inequality

$$\sum_{\beta \in J} \|(\text{id}_{\mathscr{M}_n} \otimes (\tilde{\delta}_\beta P))(A)\| \leqslant \varepsilon \sum_{\alpha \in I} \|(\text{id}_{\mathscr{M}_n} \otimes \delta_\alpha)(A)\| \tag{4.1}$$

holds, for all $n \in \mathbb{N}$, and $A \in \mathscr{M}_n \otimes \mathscr{A}$.

In particular, for $n = 1$, we get $|||PA||| \leqslant |||P|||_{\text{cb}}\, |||A|||$, i;e;, $|||P||| \leqslant |||P|||_{\text{cb}}$. Precisely as in Lemma 8 one sees that if the bound of the form (4.1) holds for all algebras $\mathscr{M}_n \otimes \mathscr{A}$, it also holds for all minimal C^*-tensor products $\mathscr{M} \otimes \mathscr{A}$ with other C^*-algebras. The crucial property of this norm is given in the following theorem.

Theorem 13. Let $P_x\colon \mathscr{A}^x \to \mathscr{B}^x$, $x \in \mathscr{L}$, be a family of completely positive unit-preserving operators between algebras with tensorable oscillation norms. Let $\mathscr{A} = \bigotimes_{x \in \mathscr{L}} \mathscr{A}^x$ and $\mathscr{B} = \bigotimes_{x \in \mathscr{L}} \mathscr{B}$ be equipped with the oscillation norm described by Proposition 9, and let $P\colon \mathscr{A} \to \mathscr{B}^x$ be defined by $P = \bigotimes_{x \in \mathscr{L}} P_x$. Then

$$|||P|||_{\text{cb}} = \sup_{x \in \mathscr{L}} |||P_x|||_{\text{cb}} \tag{4.2}$$

Proof. The inequality $|||P|||_{\text{cb}} \geqslant \sup_x |||P_x|||_{\text{cb}}$ is trivial, because the estimate (4.1) written for P and an observable $A \in \mathbb{1}_{\mathscr{M}} \otimes \mathscr{A}^x \subset \mathbb{1}_{\mathscr{M}} \otimes \mathscr{A}$ reduces to the corresponding estimate for P_x.

For the opposite inequality we have to show that, provided each P_x satisfies the estimate (4.1) with the same constant ε, i.e., $\varepsilon \geqslant |||P_x|||_{\text{cb}}$ for all $x \in \mathscr{L}$, then so does P. Consider $A \in \mathscr{M} \otimes \mathscr{A}$, and one term in the sum on the left-hand side of (4.1), say for $x \in \mathscr{L}$, and $\beta \in J^x$. We have to estimate

$$(\text{id}_{\mathscr{M}} \otimes (\tilde{\delta}_\beta^{(x)} P))(A) \\ = \left(\text{id}_{\mathscr{M}} \otimes \bigotimes_{y \neq x} P_y \otimes \text{id}_{\mathscr{A}^x} \right) \left(\text{id}_{\mathscr{M}} \otimes \bigotimes_{y \neq x} \text{id}_{\mathscr{A}^y} \otimes (\tilde{\delta}_\beta^x P_x) \right) (A)$$

The first parenthesis is a contraction because each P_y is completely positive and unital. Hence taking norms, summing over $\beta \in J^x$, and using Definition 12, we find

$$\sum_{\beta \in J^x} \|(\text{id}_{\mathscr{M}} \otimes (\tilde{\delta}_\beta^{(x)} P))(A)\| \\ \leqslant |||P_x||| \sum_{\alpha \in I^x} \left\| \left(\text{id}_{\mathscr{M}} \otimes \bigotimes_{y \neq x} \text{id}_{\mathscr{A}^y} \otimes \delta_\alpha^x \right) (A) \right\| \\ \leqslant \varepsilon \sum_{\alpha \in I^x} \|(\text{id}_{\mathscr{M}} \otimes \delta_\alpha^{(x)})(A)\|$$

The sum of these inequalities over all lattice points $x \in \mathscr{L}$ is the desired estimate, showing $|||P|||_{\text{cb}} \leqslant \varepsilon$. ∎

Hence, for a noninteracting system, it suffices to show $|||P_x|||_{\text{cb}} \leqslant \varepsilon < 1$ to conclude ergodicity from Proposition 6. However, the explicit estimate of $|||P_x|||_{\text{cb}}$ may still be a difficult problem. One way to handle it is the following *decomposition property*.

Lemma 14. In the setting of Definition 12, suppose that for each $\beta \in J$ we have a decomposition

$$\tilde{\delta}_\beta \circ P = \sum_{\alpha \in I} G_{\beta\alpha} \circ \delta_\alpha$$

with $G_{\beta\alpha}\colon \mathscr{A} \to \mathscr{B}$ linear and completely bounded, and the sum strongly convergent on $\mathscr{A}$. Then

$$|||P|||_{\text{cb}} \leqslant \sup_{\alpha \in I} \sum_{\beta \in J} \|G_{\beta\alpha}\|_{\text{cb}} \tag{4.3}$$

where $|||\cdot|||_{\text{cb}}$ is the completely bounded norm.

Proof. The proof is obvious by inserting the decompositions of $\tilde{\delta}_\beta P$ and using the triangle inequality. ∎

If $\mathscr{A}$ and $\mathscr{B}$ are finite dimensional, such decomposing maps $G_{\beta\alpha}$ always exist. In fact, the necessary and sufficient condition for an operator (here $\tilde{\delta}_\beta P$) to allow a decomposition with given δ_α is that $\ker \tilde{\delta}_\beta P \supset \bigcap_\alpha \ker \delta_\alpha$. However, the right-hand side of this inclusion is equal to $\mathbb{C}\mathbb{1}$ by part 1 of Definition 2, and, clearly, $\mathbb{C}\mathbb{1} \subset \ker \tilde{\delta}_\beta P$. We conclude that, on finite-dimensional algebras with oscillation norms defined by finitely many δ_α, we have $|||P|||_{\text{cb}} < \infty$ for all transition operators.

If $|||P|||_{\text{cb}} = 0$, we must have $\tilde{\delta}_\beta(P(A)) = 0$ for all β, so that $P(A) \equiv P_\omega(A) = \omega(A)\mathbb{1}$ for some state ω on $\mathscr{A}$. Near maps of this form we find transition operators with small oscillation norms: these could be called transition operators with *large disorder*, since in one step they wipe out nearly all memory of previous states. It is straightforward to see from the definition of the completely bounded oscillation norm that, for $0 \leqslant \lambda \leqslant 1$, the equation

$$|||(1-\lambda)\, P_\omega + \lambda P|||_{\text{cb}} = \lambda\, |||P|||_{\text{cb}} \tag{4.4}$$

holds. (In fact, the inequality $\leqslant$ already follows from the convexity of $|||\cdot|||_{\text{cb}}$). Hence, for sufficiently small λ, the ergodicity criterion, Proposition 6, applies, and $(1-\lambda)P_\omega + \lambda P$ has a unique invariant state, which will be close but not equal to ω.

In some cases, one can use the freedom of adapting the operators δ_α to the problem at hand to give a simple estimate of $|||P|||_{\text{cb}}$. An example is the following:

Lemma 15. Let $P: \mathscr{A} \to \mathscr{A}$ be a transition operator on a finite-dimensional C^*-algebra, and suppose that P is diagonalizable, i.e., it has a representation in the form

$$P = D_0 + \sum_{\alpha=1}^{N} \lambda_\alpha D_\alpha$$

with $D_\alpha D_\beta = \delta_{\alpha\beta} D_\alpha$ for $0 \leqslant \alpha, \beta \leqslant N$, and $D_0(A) = \omega(A)\mathbb{1}$ for some state ω on $\mathscr{A}$. Define an oscillation norm by setting $\delta_\alpha = D_\alpha$ for $\alpha = 1,..., N$. Then

$$|||P|||_{\text{cb}} = \max\{|\lambda_\alpha| \mid \alpha = 1,..., N\}$$

Proof. Note that the oscillation norm so defined satisfies Matsui's condition (3.9), because $\sum_{\alpha=0}^{N} D_\alpha = \text{id}$, and is hence tensorable. The lower bound $|||P|||_{\text{cb}} \geqslant \max|\lambda_\alpha|$ follows by inserting the eigenvectors into the estimate defining $|||\cdot|||_{\text{cb}}$, and the upper bound follows from Lemma 14 with $G_{\beta\alpha} = \lambda_\alpha \delta_{\alpha\beta}\text{id}$. ∎

In the classical case we may simplify the definition of $|||\cdot|||_{\mathrm{cb}}$ by considering couplings to classical systems only. For Ising systems, i.e., the case considered in ref. 2 and other principal papers on the subject, the stabilization can even be omitted entirely.

Lemma 16. Let $\mathscr{A}$, $\mathscr{B}$ be Abelian algebras with oscillation norm, and P: $\mathscr{A} \to \mathscr{B}$ a completely bounded linear map. Then:

1. $|||P|||_{\mathrm{cb}}$ is the best constant ε such that the estimate

$$\sum_{\beta \in J} \|(\mathrm{id}_{\mathscr{M}} \otimes (\tilde{\delta}_\beta P))(A)\| \leqslant \varepsilon \sum_{\alpha \in I} \|(\mathrm{id}_{\mathscr{M}} \otimes \delta_\alpha)(A)\|$$

holds for all *Abelian* C^*-algebras $\mathscr{M}$.

2. If $\mathscr{B}$ is two dimensional, or the oscillation norm on $\mathscr{B}$ is defined by a single δ, i.e., J has only one element, then $|||P|||_{\mathrm{cb}} = |||P|||$, i.e., the best constant is already achieved by taking $\mathscr{M}$ one dimensional.

Proof. As in the proof of Proposition 10, the quantum observable algebra is reduced to a classical one by evaluating in appropriate states. Let $\mathscr{A} = \mathscr{C}(X)$, $\mathscr{B} = \mathscr{C}(Y)$, and let $\mathscr{M}$ denote the C^*-algebra of bounded functions on the index set J. Suppose the bound in part 1 holds with this Abelian algebra $\mathscr{M}$, and let $A \in \mathscr{M}_n \otimes \mathscr{A}$. Thus, for each $\beta \in J$, $(\mathrm{id}_{\mathscr{M}} \otimes \tilde{\delta}_\beta P)(A)$ is a continuous $\mathscr{M}_n$-valued function on Y. By the definition (3.18) of the norm in $\mathscr{M}_n \otimes \mathscr{B}$ there is a linear functional Φ_β on $\mathscr{M}_n$ of norm $\leqslant 1$, such that

$$\begin{aligned} \|(\mathrm{id}_{\mathscr{M}_n} \otimes \tilde{\delta}_\beta P)(A)\|_{\mathscr{M}_n \otimes \mathscr{B}} &= \|\Phi_\beta \otimes \tilde{\delta}_\beta P)(A)\|_{\mathscr{B}} \\ &= \|\tilde{\delta}_\beta P(\Phi_\beta \otimes \mathrm{id}_{\mathscr{A}})(A)\|_{\mathscr{B}} \end{aligned}$$

where we used that Φ_β: $\mathscr{M}_n \to \mathbb{C}$, and $\mathbb{C} \otimes \mathscr{B} \equiv \mathscr{B}$. We now introduce the function $\hat{A} \in \mathscr{M} \otimes \mathscr{A}$, defined as $\hat{A}(\beta, x) = \Phi_\beta(A(x))$, or $\hat{A}(\beta, \cdot) = (\Phi_\beta \otimes \mathrm{id}_{\mathscr{A}})(A)$. Then the norm on the right-hand side in the above equation is smaller than

$$\sup_{\beta'} \|\tilde{\delta}_\beta P(\Phi_{\beta'} \otimes \mathrm{id}_{\mathscr{A}})(A)\|_{\mathscr{B}} = \|(\mathrm{id}_{\mathscr{M}} \otimes \tilde{\delta}_\beta P)(\hat{A})\|_{\mathscr{M} \otimes \mathscr{B}}$$

Summing over β and applying the given inequality for $\mathscr{M}$, we find

$$\sum_{\beta \in I} \|(\mathrm{id}_{\mathscr{M}_n} \otimes \tilde{\delta}_\beta P)(A)\|_{\mathscr{M}_n \otimes \mathscr{B}} \leqslant \sum_{\alpha \in I} \|(\mathrm{id}_{\mathscr{M}} \otimes \delta_\alpha)(\hat{A})\|_{\mathscr{M} \otimes \mathscr{A}}$$

and the result (1) follows, because

$$\begin{aligned}
&\|(\mathrm{id}_{\mathcal{M}} \otimes \delta_\alpha)(\hat{A})\|_{\mathcal{M}\otimes\mathcal{A}} \\
&\quad = \sup_\beta \|(\Phi_\beta \otimes \delta_\alpha)(A)\|_{\mathcal{A}} \\
&\quad \leqslant \sup_\Phi \|(\Phi \otimes \delta_\alpha)(A)\|_{\mathcal{A}} = \|(\mathrm{id}_{\mathcal{M}_n} \otimes \delta_\alpha)(A)\|_{\mathcal{A}}
\end{aligned}$$

To prove part 2, note that, on a two-dimensional algebra $\mathcal{B}$, all operators $\tilde{\delta}_\beta$ with $\tilde{\delta}_\beta(\mathbb{1}) = 0$ are proportional, so we may replace the definition of the oscillation norm by an equivalent one with $|J| = 1$. Hence the claim follows from the observation that we used part 1 only with the now one-dimensional algebra $\mathcal{M} = \mathcal{C}(J)$. ∎

However, in classical systems with more than two spin values per site we may have strict inequality $|||P|||_{\mathrm{cb}} > |||P|||$. In the following Example 17 we even have $|||P|||_{\mathrm{cb}} > 1 > |||P|||$. Hence P contracts exponentially to its fixed point, and the same is true for a noninteracting QCA with this one-site transition operator. However, for a system with nontrivial propagation maps this information is not sufficient, and only $|||P|||_{\mathrm{cb}}$ gives a bound which is independent of the propagation.

For computing $|||P|||_{\mathrm{cb}}$ Lemma 14 may be helpful even in the classical case, since then the norms $\|G_{\beta\alpha}\|_{\mathrm{cb}}$ can be replaced by ordinary norms.

We remark that with some of the modified definitions of the oscillation norms on the single-site observable algebras described at the beginning of Section 3.1, the stabilization can be avoided altogether in the classical case.[23] However, as already remarked in that context, this would be in conflict with our technique for proving the ergodicity criterion in the general quantum case.

Example 17. We take $\mathcal{A}$ as a system of two Ising spins, with its standard oscillation norm (3.1). This norm can be written as

$$\begin{aligned}
|||f||| &:= \tfrac{1}{2} \max_{\sigma_1, \sigma_2} |f(-\sigma_1, \sigma_2) - f(\sigma_1, \sigma_2)| \\
&\quad + \tfrac{1}{2} \max_{\sigma_1, \sigma_2} |f(\sigma_1, -\sigma_2) - f(\sigma_1, \sigma_2)| \\
&= \tfrac{1}{4} \left| \sum_{\sigma_1, \sigma_2} \sigma_1 f(\sigma_1, \sigma_2) \right| + \tfrac{1}{4} \left| \sum_{\sigma_1, \sigma_2} \sigma_2 f(\sigma_1, \sigma_2) \right| \\
&\quad + \tfrac{1}{2} \left| \sum_{\sigma_1, \sigma_2} \sigma_1 \sigma_2 f(\sigma_1, \sigma_2) \right|
\end{aligned}$$

where the variables σ_i are ± 1. From the second form it is obvious that the unit ball of $|||\cdot|||$ is the Cartesian product of $\mathbb{R}$ (corresponding to multiples of the identity), and an octahedron in $\mathbb{R}^3$. We now consider a transition operator $P: \mathscr{A} \to \mathscr{A}$ given by the matrix

$$P = \frac{1}{12}\begin{pmatrix} 0 & 1 & 5 & 6 \\ 0 & 0 & 2 & 10 \\ 0 & 7 & 5 & 0 \\ 8 & 0 & 0 & 4 \end{pmatrix} \tag{4.5}$$

in a basis in which the components of f are $(f(++), f(+-), f(-+), f(--))$. The oscillation norm $|||P|||$ is readily computed, by applying P to the six extreme points of the octahedral unit sphere of $|||\cdot|||$ and computing the oscillation norms of the images. The result is

$$|||P||| = 3/4 \tag{4.6}$$

We now couple the system to an additional Ising spin, denoted by σ_0, and described in the algebra $\mathscr{M} = \mathbb{C}^2$. What we have to estimate is the operator norm of $(\mathrm{id}_{\mathscr{M}} \otimes P): \mathscr{M} \otimes \mathscr{A} \to \mathscr{M} \otimes \mathscr{A}$ with respect to the norm

$$\begin{aligned} |||f|||_{\mathscr{M}\otimes\mathscr{A}} := \tfrac{1}{2} \max_{\sigma_0,\sigma_1,\sigma_2} &|f(\sigma_0, -\sigma_1, \sigma_2) - f(\sigma_0, \sigma_1, \sigma_2)| \\ + \tfrac{1}{2} \max_{\sigma_0,\sigma_1,\sigma_2} &|f(\sigma_0, \sigma_1, -\sigma_2) - f(\sigma_0, \sigma_1, \sigma_2)| \end{aligned}$$

This norm is not characterized as easily as before. The unit ball has two unbounded directions, and the compact convex set in the remaining six dimensions is bounded by 48 hyperplanes. We did not succeed in computing all the extreme points of this polytope, so we have no explicit expression for $|||\mathrm{id}_{\mathscr{M}} \otimes P|||$. However, any expression $|||(\mathrm{id}_{\mathscr{M}} \otimes P) f|||_{\mathscr{M}\otimes\mathscr{A}}$ with $|||f|||_{\mathscr{M}\otimes\mathscr{A}} = 1$ is a lower bound on $|||P|||_{\mathrm{cb}}$. Taking for f one of the 28 extreme points of the unit ball known to us, namely

$$f = (f(+++),..., f(---)) = (2, 1, 1, 0, 0, -1, -1, 0)$$

we find

$$|||P|||_{\mathrm{cb}} \geqslant |||\mathrm{id}_{\mathscr{M}} \otimes P||| \geqslant 13/12 \tag{4.7}$$

$\triangle$

4.2. Estimates in Other Norms

In this section, which is not needed later in this paper, we show how an estimate of the form (4.2) fails if we use some criteria different from oscillation norms to define contractivity of the factors. To us this is the main motivation for using oscillation norms in the first place.

Since $P_x \mathbb{1} = \mathbb{1}$, "contractivity" has to be defined in a way ignoring this known fixed point. A natural approach is to consider contractivity in the *quotient norm* of $\mathscr{A}/\mathbb{C}\mathbb{1}$, i .e.,

$$\|A\|' := \inf_{\lambda \in \mathbb{C}} \|A - \lambda \mathbb{1}\| \tag{4.8}$$

Similarly, for $P: \mathscr{A} \to \mathscr{A}$ we define

$$\|P\|' := \sup\{ \|PA\|' \mid \|A\|' \leqslant 1\} \tag{4.9}$$

and call P "contractive" if $\|P\|' < 1$. The following example shows what kind of estimate we can expect for this norm of a tensor product of transition operators.

Example 18. We consider finite classsical systems [$\mathscr{A} = \mathscr{C}(\Omega)$, Ω a finite set], for which any transition operator is of the form (2.2),

$$Pf(\omega) = \sum_{\eta} p(\omega, \eta)\, f(\eta)$$

One easily checks that

$$\|P\|' = \tfrac{1}{2} \max_{\omega, \omega'} \sum_{\eta} |p(\omega, \eta) - p(\omega', \eta)| \tag{4.10}$$

From this formula and the positivity and normalization conditions for the tensor factors, one easily finds an estimate for tensor products, namely

$$\left\| \bigotimes_{x \in \Lambda} P_x \right\|' \leqslant \sum_{x \in \Lambda} \|P_x\|' \tag{4.11}$$

Note that this estimate grows with the size of the region Λ, and becomes completely useless for infinite regions. Hence the question is whether this trivial estimate can be improved upon.

As a simple counterexample, consider an Ising spin system (i.e., $\Omega^x = \{+, -\}$ at each site x), and all factors $P_x \equiv P_1$ equal. Let χ_+ and χ_- denote the functions which are 1 on the points $+$ and $-$, respectively, and zero otherwise. P_1 is characterized by the two probabilities

$$p_{\pm} = p(\pm, +) = (P_1 \chi_+)(\pm)$$

From (4.10), $\|P_1\|' = |p_+ - p_-|$. For the products $\chi_+^A = \bigotimes_{x\in A}\chi_+$ and $P^A = \bigotimes_{x\in A} P_x$ over a finite set A of N sites we get $\|\chi_+^A\|' = 1/2$ and

$$\begin{aligned}\|P^A\chi_+^A\|' &\geqslant \tfrac{1}{2}\,|(P^A\chi_+^A)(+\cdots+) - (P^A\chi_+^A)(-\cdots-)| \\ &= \tfrac{1}{2}\,|p_+^N - p_-^N| \geqslant \tfrac{1}{2}\,|p_+ - p_-|\, N\min\{p_-, p_+\}^{N-1} \\ &= N\min\{p_-, p_+\}^{N-1}\,\|P_1\|'\,\|\chi_+^A\|'\end{aligned}$$

Picking both p_+ and p_- close to 1, we see that $\|P^A\|'$ may come arbitrarily close to $N\|P_1\|'$. △

Of course, the bound (4.11) cannot be improved upon in quantum systems either, and the norm (4.10) remains useless in infinite quantum systems as well.

Another alternative to oscillation norms, which seems plausible at first sight, is to use the observation that for norms of Hilbert space contractions with a known fixed point the right-hand side of the analog of (4.11) can be improved to a supremum. This suggests the use of the *Hilbert space norms*

$$\|A\|'_\omega = (\omega(A^*A) - |\omega(A)|^2)^{1/2} \tag{4.12}$$

on $\mathscr{A}$, and the associated operator norms for some state ω. It follows from the complete positivity of transition operators that P is a contraction with respect to this norm, provided that ω is invariant under P (i.e., $\omega\circ P = \omega$). This may not seem like a severe restriction, since we know that any transition operator admits an invariant state. We then define $\|P\|'_\omega$ as the best constant in the inequality $\|P(A)\|'_\omega \leqslant \varepsilon\,\|A\|'_\omega$. The inequality

$$\|P_1\otimes P_2\|'_{\omega_1\otimes\omega_2} \leqslant \max\,\|P_i\|'_{\omega_i}$$

holds, and it seems that we achieved our goal of finding a quantity that behaves well under composition. However, there are several drawbacks. First of all, the invariant states ω_i have to be explicitly known in order to compute any norm. Second, the ergodicity statement one gets from the inequality $\|P\|'_\omega < 1$ is rather weak: it allows no conclusion about states of the infinite system which are singular with respect to ω. Perhaps the most severe restriction, however, is that the propagation operators introducing interaction into the QCA setting by mixing different cells also fail to be contractions with respect to this norm, so the approach based on (4.12) seems to be limited to the trivial, noninteracting case.

A similar criticism applies to the idea to use the *spectral radius* of P as an operator on $\mathscr{A}/\mathbb{C}\mathbb{1}$, denoted by $\sigma'(P)$. Again, we have equality $\sigma'(P_1\otimes P_2) = \max_i \sigma'(P_i)$, but we have no control over this quantity for a product of two transition operators, which we need to introduce interaction (see the proof of Theorem 19). As an elementary example consider, as in Example 17, a classical system with two subcell types of one Ising spin each, with the transition operator

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

In a system consisting of two cells we choose the propagation map $d(1;\cdot)$ for the first subcell type to be the identity, and the map $d(2;\cdot)$ for the second subcell type to be the flip. Then P_1 contracts exponentially with rate 1/2 to its invariant state, which is the pure state on the configuration $(+-)$. However, the total transition operator P has three invariant states.

Other conceivable alternatives are the *sup-oscillation norms* introduced in (3.3), taken now not only as a way to define the oscillation norm in the subcells, but as a principle to construct the total oscillation norm. Defining the operator norm $\overline{|||\cdot|||}_{cb}$ in analogy to Definition 12, it is easy to show the analog of Theorem 13. However, this does not suffice to give an ergodicity criterion, since the estimate (3.6) fails for such norms, and consequently $\overline{|||\cdot|||}$-Cauchy sequences need not converge in $\mathscr{A}$. A simple example demonstrating these claims is the sequence of averages of Ising spins over an increasing sequence of regions. The sup-oscillation norm of such averages goes to zero like the inverse number of sites in the average, but, of course, the sequence of averages is not convergent in the quasi-local algebra.

5. APPLICATIONS TO CELLULAR AUTOMATA

5.1. Ergodicity

In order to apply the results of the previous section to interacting QCAs, we need tensorable oscillation norms on the algebras $\mathscr{B}^s$ belonging to each subcell type using, say, operators $\delta^s_\alpha = \mathscr{B}^s \to \mathscr{B}^s$, $\alpha \in I^s$. Then by Proposition 9 we have tensorable oscillation norms on each $\mathscr{A}^x$, and consequently on the algebra $\mathscr{A}^{\mathscr{L}}$ of the whole system. By Proposition 6 ergodicity follows from the estimate $|||P||| < 1$ for the total transition operator P. Thus we arrive at the following criterion.

Theorem 19. Let a quantum cellular automaton be given according to Definition 1, and suppose that each $\mathscr{B}^s$ is equipped with a tensorable oscillation norm. Then

$$|||P|||_{\mathrm{cb}} = |||P_1|||_{\mathrm{cb}}$$

Consequently, if $|||P_1|||_{\mathrm{cb}} < 1$, the QCA is ergodic, i.e., there is a unique P-invariant state ρ on $\mathscr{A}^{\mathscr{L}}$.

Proof. Let $\tilde{P} = \bigotimes_{x \in \mathscr{L}} P_1$ be the infinite tensor product of the operators P_1 acting in each $\mathscr{A}^x$ separately. This is also the total transition operator of the QCA with the same P_1, but each $d(s; \cdot)$ equal to the identity. Consider also the automorphism $D: \mathscr{A}^{\mathscr{L}} \to \mathscr{A}^{\mathscr{L}}$ which takes $\mathscr{B}^{(x,s)}$ into $\mathscr{B}^{d(s;x),s}$. Then

$$P = D\tilde{P}$$

According to Theorem 13, $|||\tilde{P}_1|||_{\mathrm{cb}} = |||P_1|||_{\mathrm{cb}}$. Moreover, since the oscillation norm on $B^{d(s;x),s}$ is defined by the same operators δ^s_α as in $\mathscr{B}^{x,s}$, D is a $|||\cdot|||$-isometry, and $|||D|||_{\mathrm{cb}} = 1$. Hence $|||P|||_{\mathrm{cb}} = |||D|||\tilde{P}|||_{\mathrm{cb}} = |||P_1|||_{\mathrm{cb}}$. ∎

Note that similarly to the noninteracting case, the criterion $|||P_1|||_{\mathrm{cb}} < 1$ is a condition of "large disorder" which is satisfied as soon as P_1 is sufficiently close to a map of the form $P_1(A) = \omega(A)\mathbb{1}$ [compare (4.4)]. A remarkable feature of this criterion is that it does not depend on the propagation maps $d(s; \cdot)$ which distinguish an interacting QCA from a noninteracting one.

One might expect from Lemma 15 that, with a suitable choice of oscillation norms, $|||P_1|||_{\mathrm{cb}}$ can be made equal to the largest modulus of eigenvalues of P_1 apart from 1. However, this is not the case, since it is crucial for the proof of Theorem 19 that each oscillation operator δ^s_α acts in only one subcell $\mathscr{B}^s$, and so the propagation automorphism D becomes a $|||\cdot|||$-isometry. Clearly, this property cannot be expected of the eigenprojections of P_1. Still, the second largest modulus of eigenvalues of P_1 is always a lower bound to $|||P_1|||_{\mathrm{cb}}$.

5.2. The Classical Case

Since in the usual definition of PCAs no subcell decomposition is used, it is not obvious how the classical results[2,1] can be subsumed under Theorem 19. In this section we show how this can be done, pointing at the same time to a possible generalization of Definition 1.

The basic operator P_1 defining the one-site transition probability of a PCA [see (2.2)] maps the one-site algebra $\mathscr{A}$ into the tensor product

$\tilde{\mathscr{A}} = \mathscr{A}^{\otimes n}$, where n is the number of cells influencing the state in a single cell of the second generation. n is often finite, but we do not need this fact. We will construct a QCA whose one-site algebra is $\tilde{\mathscr{A}}$, with all subcell algebras $\mathscr{B}^s$ isomorphic to $\mathscr{A}$. Note that all these algebras are now commutative, so the "Q" in QCA only refers to the fulfilment of Definition 1.

The total PCA transition operator P [see (2.3)] can be decomposed into three factors: the first is simply the infinite tensor product $P_1^{\mathscr{L}}$ of the operator P_1 mapping $\mathscr{A}^{\mathscr{L}}$ to $\tilde{\mathscr{A}}^{\mathscr{L}}$. The information about the cell y to which a subcell (x, s) in the latter algebra belongs in the next time step is encoded in propagation maps $d(s; \cdot): \mathscr{L} \to \mathscr{L}$ as before, i.e., $y = d(s; x)$. This defines an automorphism D of $\tilde{\mathscr{A}}^L$ as in the proof of Theorem 19. The final step is the sitewise application of the multiplication map $\mathbf{M}: \tilde{\mathscr{A}} \to \mathscr{A}$, defined by

$$\mathbf{M}(f_1 \otimes \cdots \otimes f_n) = \prod_{i=1}^{n} f_i \tag{5.1}$$

Hence we get the factorization

$$P = \mathbf{M}^{\mathscr{L}} D P_1^{\mathscr{L}} \tag{5.2}$$

where $\mathbf{M}^{\mathscr{L}}$ denotes the tensor product of the copies of $\mathbf{M}$ acting at each site.

It is precisely the use of the multiplication map $\mathbf{M}$, that is the specifically classical element in this construction. $\mathbf{M}$ is also called the nth-order *diagonal* of the algebra $\mathscr{A}$ because, writing $\mathscr{A} \cong \mathscr{C}(\Omega)$ and $\mathscr{A}^{\otimes n} = \mathscr{C}(\Omega^n)$, as we may for an Abelian algebra, we have

$$\mathbf{M}f(x) = f(x, x, ..., x) \tag{5.3}$$

Clearly, $\mathbf{M}$ is a *-homomorphism, and $\mathbf{M}(\mathbb{1} \otimes ... \otimes f \otimes ... \otimes \mathbb{1}) = f$, for all positions of the factor f in the tensor product. The existence of the diagonal characterizes Abelian algebras: if a homomorphism $\mathbf{M}$ of this description exists in a C^*-algebra $\mathscr{A}$, the commutativity of the tensor multiplication implies the commutativity of multiplication. The adjoint of the diagonal map is a "state duplication map" reproducing, from a state on $\mathscr{A}$, n copies of $\mathscr{A}$ in the same state. Its nonexistence in the quantum case is the basis of "quantum cryptography" (see ref. 24 and references cited there). Of course, we can formally define $\mathbf{M}$ by (5.1), even in the noncommutative case. However, if $\mathscr{A} = \mathscr{M}_d$, $n = 2$, and Φ is the unitary permutation operator exchanging the two factors, $\|\mathbf{M}(\Phi)\| = n$, i.e., $\|\mathbf{M}\| \geqslant n$. Hence, on an infinite-dimensional algebra, $\mathbf{M}$ is typically unbounded. But even in the finite-dimensional case, $\|\mathbf{M}\| > 1$ makes the definition of the infinite tensor product $\mathbf{M}^{\mathscr{L}}$ in (5.2) impossible.

It is now easy to modify (5.2) so that we get a QCA in the sense of Definition 1. Its transition operator is

$$\tilde{P} := D P_1^{\mathscr{L}} \mathbf{M}^{\mathscr{L}} \tag{5.4}$$

One easily verifies that this is the QCA with one-site transition operator

$$\tilde{P}_1: \quad \tilde{\mathscr{A}} \to \tilde{\mathscr{A}}$$

$$\tilde{P}_1 = P_1 \mathbf{M}$$

With $J: \mathscr{A} \to \tilde{\mathscr{A}}$, defined as $Jf = f \otimes \mathbb{1}^{\otimes (n-1)}$, we have $\mathbf{M}J = \mathrm{id}_{\mathscr{A}}$, and hence

$$P^N = \mathbf{M}^{\mathscr{L}} \tilde{P}^N J^{\mathscr{L}} \tag{5.5}$$

for every power $N \geqslant 0$, i.e., the PCA can be recovered completely from the QCA picture.

For estimating the contraction rates of these operators we need the following lemma.

Lemma 20. Let $\mathscr{A} = \mathscr{C}(\Omega)$ be a finite dimensional Abelian C^*-algebra with an oscillation norm defined by operators δ_α of the form

$$\delta_\alpha f(\sigma) = c_\alpha (f(a_\alpha(\sigma)) - f(\sigma))$$

where $c_\alpha \in \mathbb{R}$ and $a_\alpha: \Omega \to \Omega$. Then, for every n, the nth-order diagonal $\mathbf{M}$ satisfies the estimate $|||\mathbf{M}|||_{\mathrm{cb}} \leqslant 1$.

Proof. $\mathscr{M}_d \otimes \mathscr{A}$ can be identified with the algebra of $\mathscr{M}_d$-valued functions on Ω equipped with the norm $\|g\| = \sup_\sigma \|g(\sigma)\|$. Then, for $f \in \mathscr{M}_d \otimes \mathscr{A}^{\otimes n}$ and $\sigma' = a_\alpha(\sigma)$, the expression

$$F_k = f(\underbrace{\sigma', \ldots, \sigma'}_{k \text{ times}}, \underbrace{\sigma, \ldots, \sigma}_{n-k \text{ times}}) - f(\underbrace{\sigma', \ldots, \sigma'}_{k-1 \text{ times}}, \underbrace{\sigma, \ldots, \sigma}_{n-k+1 \text{ times}})$$

is bounded in the norm of $\mathscr{M}_d$ by $|c_\alpha^{-1}| \cdot \|(\mathrm{id}_{\mathscr{M}_d} \otimes \delta_\alpha^{(k)}) f\|$. Hence

$$\begin{aligned} \|((\mathrm{id}_{\mathscr{M}_d} \otimes \delta_\alpha \mathbf{M}) f)(\sigma)\| &= |c_\alpha| \cdot \|f(\sigma', \ldots, \sigma') - f(\sigma, \ldots, \sigma)\| \\ &= |c_\alpha| \cdot \left\| \sum_{k=1}^{n-1} F_k \right\| \\ &\leqslant \sum_{k=1}^{n-1} \|(\mathrm{id}_{\mathscr{M}_d} \otimes \delta_\alpha^{(k)}) f\| \end{aligned}$$

Hence, after taking the supremum over σ, summing over α, and using the definition (3.13) of the canonical oscillation norm on $\mathscr{A}^{\otimes n}$, we get (4.1) with $\varepsilon = 1$. The constant cannot be better than 1, because $\mathbf{M}J = \mathrm{id}_{\mathscr{A}}$, and, obviously $|||Jf||| = |||f|||$ for all f. ∎

Hence, from the factorization (5.5) we get $|||P|||_{\mathrm{cb}} \leqslant |||\mathbf{M}^{\mathscr{L}}|||_{\mathrm{cb}}\, |||\tilde{P}|||_{\mathrm{cb}}\, |||J^{\mathscr{L}}|||_{\mathrm{cb}} \leqslant |||\tilde{P}|||_{\mathrm{cb}}$. From (5.4) and because D is an oscillation norm isometry, $|||\tilde{P}|||_{\mathrm{cb}} \leqslant |||P_1^{\mathscr{L}}|||_{\mathrm{cb}}$. Because $P_1^{\mathscr{L}} = D^{-1}\tilde{P}J^{\mathscr{L}}$, the last inequality is actually an equality. By Theorem 13, $|||P_1^{\mathscr{L}}|||_{\mathrm{cb}} = |||P_1|||_{\mathrm{cb}}$. Summing up these estimates, we have

$$|||P|||_{\mathrm{cb}} \leqslant |||\tilde{P}|||_{\mathrm{cb}} = |||P_1|||_{\mathrm{cb}}$$

This is exactly the bound given in ref. 2. When comparing these results, however, note that Proposition 6 gives $\|P^N(A) - \rho(A)\mathbb{1}\| \leqslant 2\varepsilon^n |||A|||$, without the superfluous factor $(1-\varepsilon)^{-1}$ present in ref. 2.

Finally, we wish to point out that (5.4) also points to a possible generalization of the notion of QCA, which does not use subcell decompositions: we only have to replace $\mathbf{M}$ by some completely positive unital operator from $\mathscr{A}^{\otimes n} \to \mathscr{A}$. For such systems our method for obtaining oscillation norm estimates would apply unchanged, but they would no longer satisfy the condition of commuting ranges (see Section 2). In this sense the cells would no longer be "independently updated."

5.3. Decay of Correlations in the Invariant State

We assume now that the ergodicity criterion $|||P_1|||_{\mathrm{cb}} < 1$ holds. What can be said about the unique invariant state ρ to which P contracts? It is clear that in the noninteracting case [i.e., $d(s; \cdot) = \mathrm{id}$], but also if $P_1(A) = \omega(A)\mathbb{1}$ (i.e, $|||P_1|||_{\mathrm{cb}} = 0$), ρ will be a product state. Therefore, it is reasonable to expect that if $|||P_1|||_{\mathrm{cb}}$ is small, we should obtain a state with good clustering properties. Moreover, in contrast to the ergodicity criterion, the propagation maps $d(s; \cdot)$ should enter the estimate for the correlation functions.

We will first describe the relevant geometric properties of the $d(s; \cdot)$ For $\Lambda \subset \mathscr{L}$, we will set

$$d_S(\Lambda) = \{d(s; x) \mid s \in S, x \in \Lambda\} \subset \mathscr{L}$$

This is the set of cells to which the interaction can spread from some site in Λ in one step. Similarly, we define $d_S^n(\Lambda)$ as the nth iterate of d_S. For $\Lambda_1, \Lambda_2 \subset \mathscr{L}$ we define the *correlation distance* as

$$c(\Lambda_1, \Lambda_2) \max\{n \in \mathbb{N} \mid d_S^n(\Lambda_1) \cap d_S^n(\Lambda_2) = \varnothing\}$$

i.e., as the last time step under which the two regions remain independent. When the maps $d(s; \cdot)$ are translations, it is clear that for large separation parameters r, the correlation distance $c(\Lambda_1, \Lambda_2 + r\vec{e})$ will asymptotically be proportional to r, but with a constant depending on the direction $\vec{e}$. In this sense the following proposition gives exponential clustering with a rate depending both on the direction and on $|||P_1|||_{\mathrm{cb}}$.

Proposition 21. Let a quantum cellular automaton be given according to Definition 1, and suppose that, with respect to some choice of tensorable oscillation norms on each $\mathscr{B}^s$, the ergodicity criterion $|||P_1|||_{\mathrm{cb}} < 1$ is satisfied. Let ρ denote the unique state such that $\rho \circ P = \rho$. Let

$$A_1 \in \mathscr{A}^{\Lambda_1} = \bigotimes_{x \in \Lambda_1} \mathscr{A}^x, \qquad A_2 \in \mathscr{A}^{\Lambda_2} = \bigotimes_{x \in \Lambda_2} \mathscr{A}^x$$

where $\Lambda_1 \cap \Lambda_2 = \varnothing$ are disjoint finite subsets of $\mathscr{L}$. Then:

1. $P^k(A_1) \subset \mathscr{A}^{d_s^k(\Lambda_1)}$ for all $k \in \mathbb{N}$.
2. $P^k(A_1 \otimes A_2) = P^k(A_1) \otimes P^k(A_2)$ for all $k \leqslant c(\Lambda_1, \Lambda_2)$.
3. $|\rho(A_1 \otimes A_2) - \rho(A_1)\,\rho(A_2)|$
   $\leqslant 2(|||P_1|||_{\mathrm{cb}})^{c(\Lambda_1, \Lambda_2)}(|||A_1||| \cdot \|A_2\| + \|A_1\| \cdot |||A_2|||)$.

Proof. The first two statements are obvious from Definition 1 for $k = 1$, and follow for other k by induction. Then using Proposition 6 and Theorem 19, we get, for all $k \leqslant c(\Lambda_1, \Lambda_2)$, the estimate

$$\begin{aligned}
&|\rho(A_1 \otimes A_2) - \rho(A_1)\,\rho(A_2)| \\
&\quad = \|\rho(A_1 \otimes A_2)\,\mathbb{1} - \rho(A_1)\,\rho(A_2)\,\mathbb{1}\| \\
&\quad \leqslant \|\rho(A_1 \otimes A_2)\,\mathbb{1} - P^k(A_1 \otimes A_2)\| + \|P^k(A_1 \otimes A_2) - P^k(A_1) \otimes P^k(A_2)\| \\
&\qquad + \|P^k(A_1) - \rho(A_1)\,\mathbb{1}\| \cdot \|P^k(A_2)\| + |\rho(A_1)| \cdot \|P^k(A_2) - \rho(A_2)\,\mathbb{1}\| \\
&\quad \leqslant (|||P_1|||_{\mathrm{cb}})^k\, |||A_1 \otimes A_2||| + (|||P_1|||_{\mathrm{cb}})^k\, \|A_2\| \cdot |||A_1||| \\
&\qquad + (|||P_1|||_{\mathrm{cb}})^k\, \|A_1\| \cdot |||A_2|||
\end{aligned}$$

where at the last step we used that $\|P^k(A_2)\| \leqslant \|A_2\|$. The result then follows from Eq. (3.19). ∎

APPENDIX. COMPLETE BOUNDEDNESS

To motivate the necessity of considering complete positivity and complete boundedness of operators on noncommutative C^*-algebras we consider a standard example: the operator $P\colon \mathcal{M}_n \to \mathcal{M}_n$ of transposition on the algebra of $n\times n$ matrices. This preserves positivity and the identity element, and therefore seems to be a candidate for a transition operator. However, positivity and the norm bound $\|P\| \leqslant 1$ both get lost if we consider the system as a subsystem of a larger one with observable algebra, say $\mathcal{M}_n \otimes \mathcal{M}_n$. Then $(\mathrm{id}_{\mathcal{M}_n} \otimes P)$ takes the unitary flip operator $\Phi = \sum_{ij} |ij\rangle\langle ji|$ into n times the one-dimensional projection $p=(1/n)\sum_{ij} |ii\rangle\langle jj|$. Hence $\|\mathrm{id}_{\mathcal{M}_n} \otimes P\| \geqslant n$, and $P(\mathbb{1}-\Phi) = \mathbb{1} - np$ is not positive, although $(\mathbb{1}-\Phi)$ is. Clearly, $\mathrm{id}_{\mathcal{M}_n} \otimes P$ is no longer a transition operator, although there is no interaction with the "innocent bystander" system described in $\mathcal{M}_n$.

In order to exclude such phenomena one defines a linear operator $P\colon \mathcal{A} \to \mathcal{B}$ between C^*-algebras to be *completely positive*[20, 25] if $\mathrm{id}_{\mathcal{M}_n} \otimes P$ is positive for all n or, equivalently,[20] if for any choice of n-tuples $a_1,\ldots, a_n \in \mathcal{A}$ and $b_1,\ldots, b_n \in \mathcal{B}$ the operator $\sum_{ij} b_i^* P(a^*_i a_j) b_j$ is positive. P is said to be *completely bounded* if $\|\mathrm{id}_{\mathcal{M}_n} \otimes P\|$ is bounded by a constant independent of n. For such operators we define the *completely bounded norm* as

$$\|P\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|\mathrm{id}_{\mathcal{M}_n} \otimes P\|$$

If $\|P\|_{\mathrm{cb}} \leqslant 1$, P is called a *complete contraction*. The appearance of the matrix algebras $\mathcal{M}_n$ in these definitions is solely a matter of convenience: these definitions imply the corresponding statements with $\mathcal{M}_n$ replaced by an arbitrary C^*-algebra $\mathcal{M}$ (see the proof of Lemma 8 for a very similar argument). For checking complete positivity or boundedness it is often useful to consider $\mathcal{M}_n \otimes \mathcal{A}$ as the *-algebra of $n\times n$-matrices with entries in $\mathcal{A}$. When $\mathcal{B}$ is a finite-dimensional algebra containing as direct summands at most the $k\times k$ matrices, it suffices to verify complete positivity, or to compute $\|P\|_{\mathrm{cb}}$, in $\mathcal{M}_n \otimes \mathcal{A}$ with $n=k$.[26] In particular, all operators between finite-dimensional C^*-algebras are completely bounded.

Basic examples of completely positive maps are *-homomorphisms, maps of the form $A \mapsto V^*AV$, and all positive maps with either $\mathcal{A}$ or $\mathcal{B}$ Abelian, which includes all states. Completely positive operators are completely bounded, and when $\mathbb{1} \in \mathcal{A}$ and P is completely positive, we have $\|P(\mathbb{1})\| = \|P\| = \|P\|_{\mathrm{cb}}$.[25]. The fundamental structure theorem for completely positive maps is the Stinespring dilation theorem,[27] stating that every completely positive $P\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ can be decomposed in an essentially unique way into $P(A) = V^*\pi(A)V$, where $\pi\colon \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a *-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{K}$, and $V\colon \mathcal{H} \to \mathcal{K}$ is a bounded operator.

Basic examples of complete contractions are differences $P = P_+ - P_-$ of completely positive maps with $\|P_+ + P_-\| \leqslant 1$, and multiplication operators $A \mapsto MA$ where $\|M\| \leqslant 1$. Results analogous to the Stinespring dilation are also available for completely bounded maps. However, the uniqueness is typically lost. Thus any complete contraction $P: \mathscr{A} \to \mathscr{B}(\mathscr{H})$ with $\mathscr{A} \ni \mathbb{1}$ can be decomposed as $P(A) = V^*{}_1 \pi(A) V_2$, with π a *-representation of $\mathscr{A}$ and V_1 and V_2 isometrics. Essentially the same statement is that every complete contraction $P: \mathscr{A} \to \mathscr{B}(\mathscr{H})$ can be realized as the off-diagonal corner of a completely positive map,[25] i.e., there is a unit-preserving completely positive map $\tilde{P}: \mathscr{M}_2 \otimes \mathscr{A} \to \mathscr{M}_2 \otimes \mathscr{B}(\mathscr{H})$ such that

$$\tilde{P}\left(\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & P(A) \\ 0 & 0 \end{pmatrix}$$

Every completely bounded operator $P: \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is a linear combination of completely positive ones. If, moreover, P is hermitian [i.e., $P(A^*) = P(A)^*$], one can find a completely positive P_+ with $\|P_+\|_{\text{cb}} = \|P\|_{\text{cb}}$ such that $P_+ \pm P$ are completely positive.[28, 25] The same statement holds when $\mathscr{B}(\mathscr{H})$ is replaced by an arbitrary injective C^*-algebra,[28] but fails in general. Since finite-dimensional algebras are injective, this covers the applications of this result in connection with Lemma 14.

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